COLLOQUIUM MATHEMATICUM

VOL. LVIII 1989 FASC. 1

# *A RANDOM NONLINEAR MULTIVALUED EVOLUTION EQUATION IN HILBERT SPACE*

BY

DIMITRIOS KRAVVARITIS (ATHENS)
AND NIKOLAOS S. PAPAGEORGIOU (DAVIS, CALIFORNIA)

**1. Introduction.** In many instances in mathematical modelling, due to unpredictable variations of the data or even due to our ignorance, we are led to random differential systems. This is very nicely exemplified by the works of Bharucha-Reid [5], Soong [18], and Tsokos and Padgett [19].

In the area of partial differential equations, Kampé de Fériet [10], [11] was the first to study the random heat equation, with the randomness entering through the initial value of the problem. Later Samuels [17] considered the same problem with random boundary condition. More recently Becus [3], [4] combined random initial and boundary conditions with random source (random forcing term), while Chow [8], in connection with the problem of wave propagation in random media, had an equation with random operators.

In this paper we unify and extend the above-mentioned results by examining the following general random, nonlinear multivalued evolution equation involving convex subdifferentials:

$$(*) \qquad -\dot{x}(\omega, t) \in \partial\varphi(\omega, x(\omega, t)) + F(\omega, t, x(\omega, t)), \qquad x(\omega, 0) = x_0(\omega).$$

By a *random strong solution* of $(*)$ we understand a stochastic process $x\colon \Omega \times T \to X$ with absolutely continuous realizations such that, for every $\omega \in \Omega$, $x(\omega, \cdot)$ is a strong solution of $(*)$, i.e., for fixed $\omega \in \Omega$, $x(\omega, \cdot)$ satisfies $(*)$ a.e. Deterministic equations of this form and with single valued perturbations were studied by Brezis [6] and Barbu [2].

In the remaining part of this section we will recall some definitions from the theory of multifunctions that we will need in the sequel. So let $(\Omega, \Sigma, \mu)$ be a complete probability space, and $X$ a separable Banach space. By $P_{f(c)}(X)$ we will denote the nonempty closed (convex) subsets of $X$. A multifunction $F\colon \Omega \to P_f(X)$ is said to be *measurable* if, for all $z \in X$,

$$\omega \to d(z, F(\omega)) = \inf\{\|z-x\|\colon x \in F(\omega)\}$$

is measurable. By $S_F^p$, $1 \leqslant p \leqslant \infty$, we will denote the $L^p(X)$-selectors of $F(\cdot)$, i.e.,

$$S_F^p = \{f \in L^p(X)\colon f(\omega) \in F(\omega) \ \mu\text{-a.e.}\}.$$

This set may be empty. It is nonempty if and only if

$$\omega \to \inf\{\|z\|: z \in F(\omega)\} \in L^1.$$

Also, let $\{A_n\}_{n \geq 1} \subseteq 2^X \backslash \{\emptyset\}$. Then the *weak limit superior* of the $A_n$'s is defined to be the set

$$\text{w-}\overline{\lim} A_n = \{x \in X: x = \text{w-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, k \geq 1\}.$$

If $Y$, $Z$ are Hausdorff topological spaces, we say that a multifunction $F: Y \to 2^Z \backslash \{\emptyset\}$ is *upper semicontinuous* (u.s.c.) if and only if, for all $U \subseteq Z$ open, $F^+(U) = \{y \in Y: F(y) \subseteq U\}$ is open in $Y$. Furthermore, we say that $F(\cdot)$ is *closed* if

$$\text{Gr}F = \{(y, z) \in Y \times Z: z \in F(y)\}$$

is closed in $Y \times Z$. From [9] we know that upper semicontinuity implies closedness of $F(\cdot)$. The converse is not true in general.

A function $\varphi: \Omega \times X \to \bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$ is a *normal integrand* if and only if $(\omega, x) \to \varphi(\omega, x)$ is measurable and, for all $\omega \in \Omega$, $x \to \varphi(\omega, x)$ is l.s.c. and proper. Using von Neumann's projection theorem (see [16], Theorem 4), we can see that the multifunction

$$\omega \to \text{epi}\, \varphi(\omega, \cdot) = \{(x, \lambda) \in X \times \boldsymbol{R}: \varphi(\omega, x) \leqslant \lambda\}$$

is closed valued and measurable. If, in addition, $\varphi(\omega, \cdot)$ is convex, then we say that $\varphi$ is a *convex normal integrand.*

Finally, from nonsmooth analysis recall that for a proper function $\varphi: X \to \bar{\boldsymbol{R}}$ the *subdifferential* at $x$ is defined by

$$\partial\varphi(x) = \{x^* \in X^*: (x^*, y - x) \leqslant \varphi(y) - \varphi(x) \text{ for all } y \in X\},$$

while $\varphi(\cdot)$ is said to be *inf-compact* if, for all $\lambda \in \boldsymbol{R}$, $\{x \in X: \varphi(x) \leqslant \lambda\}$ is compact.

**2. The Theorem.** The mathematical setting is the following. Let $(\Omega, \Sigma, \mu)$ be a complete probability space, $T = [0, b]$ a bounded closed interval in $\boldsymbol{R}_+$, and $X$ a separable Hilbert space. We will make the following hypotheses:

$H(\varphi)$: $\varphi: \Omega \times X \to \boldsymbol{R}$ is a convex normal integrand, which is inf-compact in the $x$-variable and the multifunction

$$D(\omega) = D(\partial\varphi(\omega, \cdot)) = \{x \in X: \partial\varphi(\omega, x) \neq \emptyset\}$$

has a measurable selector.

$H(F)$: $F: \Omega \times T \times X \to P_{fc}(X)$ is a multifunction such that

(i) $(\omega, t, x) \to F(\omega, t, x)$ is measurable;

(ii) for every $(\omega, t) \in \Omega \times T$, the multifunction $x \to F(\omega, t, x)$ is sequentially closed in $X \times X_w$;

(iii) $|F(\omega, t, x)| = \sup\{\|z\|: z \in F(\omega, t, x)\} \leqslant \alpha(\omega, t) + b(\omega, t)\|x\|$ a.e. for every $\omega \in \Omega$, with $\alpha(\cdot, \cdot)$ and $b(\cdot, \cdot)$ measurable and $\alpha(\omega, \cdot), b(\omega, \cdot) \in L^2$;

$H_0$: $x_0$: $\Omega \to X$ is measurable.

We have the following existence result concerning problem $(*)$:

THEOREM. *If hypotheses $H(\varphi)$, $H(F)$ and $H_0$ hold, then $(*)$ admits a random strong solution.*

Proof. Let $z(\cdot)$ be a measurable selector of $D(\cdot)$ and let $v(\cdot)$ be a measurable selector of $\omega \to \partial\varphi(\omega, z(\omega))$. The latter is possible since the subdifferential multifunction is measurable. Set

$$\hat{\varphi}(\omega, x) = \varphi(\omega, x) - \varphi(\omega, z(\omega)) - (v(\omega), x - z(\omega)).$$

Then it is easy to check that $(*)$ is equivalent to

$$-\dot{x}(\omega, t) \in \partial\hat{\varphi}(\omega, x(\omega, t)) + F(\omega, t, x(\omega, t)) + v(\omega), \qquad x(\omega, 0) = x_0(\omega).$$

So there is no loss of generality in assuming that

$$\min\{\varphi(\omega, x)\colon x \in X\} = \varphi(\omega, z(\omega)) = 0.$$

Therefore

$$(z(\omega), 0) \in \operatorname{Gr}\partial\varphi(\omega, \cdot) \quad \text{for all } \omega \in \Omega.$$

Next let us determine an a priori bound for the solutions of $(*)$. Thus, if $x(\cdot,\cdot)$ is a random strong solution, it is automatically an integral solution too, and so it satisfies (see [6], p. 64)

$$\|x(\omega, t) - x\|^2 \leqslant \|x_0(\omega) - x\|^2 + 2\int_0^t (h(\omega)(s) - y, x(\omega, s) - x)\, ds$$

for every $\omega \in \Omega$, $(x, y) \in \operatorname{Gr}\partial\varphi(\omega, \cdot)$ and for some $h(\omega) \in S^2_{F(\omega,\cdot,x(\omega,\cdot))}$. Let $(x, y) = (z(\omega), 0)$. Then

$$\|x(\omega, t) - z(\omega)\|^2 \leqslant \|x_0(\omega) - z(\omega)\|^2 + 2\int_0^t (h(\omega)(s), x(\omega, s) - z(\omega))\, ds,$$

so

$$\|x(\omega, t) - z(\omega)\|^2 \leqslant \|x_0(\omega) - z(\omega)\|^2 + 2\int_0^t \|h(\omega)(s)\| \cdot \|x(\omega, s) - z(\omega)\|\, ds.$$

Invoking Lemma A.5, p. 157, of [6], we get

$$\|x(\omega, t) - z(\omega)\| \leqslant \|x_0(\omega) - z(\omega)\| + \int_0^t \|h(\omega)(s)\|\, ds \quad \text{for all } \omega \in \Omega,\ t \in T.$$

Recalling that $h(\omega)(\cdot) \in S^1_{F(\omega,\cdot,x(\omega,\cdot))}$, $\omega \in \Omega$, and using the growth hypothesis $H(F)$ (iii), we have

$$\|x(\omega, t) - z(\omega)\| \leqslant \|x_0(\omega) - z(\omega)\| + \int_0^t [\alpha(\omega, s) + b(\omega, s)\|x(\omega, s)\|]\, ds,$$

and by Gronwall's inequality we get

$$\|x(\omega, t)\| \leqslant K(\omega)\exp\|b(\omega, \cdot)\|_1 = M(\omega) \quad \text{for all } \omega\in\Omega,\ t\in T,$$

where

$$K(\omega) = \|x_0(\omega)-z(\omega)\| + \|z(\omega)\| + \|\alpha(\omega, \cdot)\|_1.$$

Then consider the multifunction $\hat{F}$: $\Omega\times T\times X\to P_{fc}(X)$ defined by

$$\hat{F}(\omega, t, x) = \begin{cases} F(\omega, t, x) & \text{if } \|x\| \leqslant M(\omega), \\ F\left(\omega, t, \dfrac{M(\omega)x}{\|x\|}\right) & \text{if } \|x\| > M(\omega). \end{cases}$$

From this definition we see that

$$\hat{F}(\omega, t, x) = F(\omega, t, p(\omega, x)),$$

where, for every $\omega\in\Omega$, $p(\omega, \cdot)$ is the $M(\omega)$-radial retraction. Note that $p(\omega, x)$ is measurable in $\omega$ and Lipschitz in $x$. Hence $(\omega, t, x)\to\hat{F}(\omega, t, x)$ is measurable, while for $x_n \xrightarrow{s} x$ we have

$$p(\omega, x_n)\xrightarrow{s} p(\omega, x),$$

and since, by $H(F)$ (ii), $F(\omega, t, \cdot)$ is closed and $B(0, M(\omega)) = \{x\in X\colon \|x\| \leqslant M(\omega)\}$ is metrizable in the w-topology, we have

$$\text{w-}\overline{\lim}\, F(\omega, t, p(\omega, x_n)) \subseteq F(\omega, t, p(\omega, x)) \Rightarrow \text{w-}\overline{\lim}\,\hat{F}(\omega, t, x_n) \subseteq \hat{F}(\omega, t, x).$$

Furthermore, $|\hat{F}(\omega, t, x)| \leqslant \alpha(\omega, t)+M(\omega)b(\omega, t) = \gamma(\omega, t)$. Clearly, $\gamma(\cdot,\cdot)$ is jointly measurable and, for all $\omega\in\Omega$, $\gamma(\omega, \cdot)\in L^2$. Consider the set

$$B(\gamma)(\omega) = \{h\in L^2(X)\colon \|h(t)\| \leqslant \gamma(\omega, t) \text{ a.e.}\}.$$

We claim that $\omega\to B(\gamma)(\omega)$ is a measurable multifunction. To see this let

$$C(\omega, t) = \{x\in X\colon \|x\| \leqslant \gamma(\omega, t)\}.$$

Since $\gamma(\cdot,\cdot)$ is jointly measurable, so is the multifunction $(\omega, t)\to C(\omega, t)$ and, for all $\omega\in\Omega$, $C(\omega, \cdot)$ is $L^2$-bounded. Now note that $B(\gamma)(\omega) = S^2_{C(\omega,\cdot)}$. Then for every $k\in L^2(X)$ we have

$$\begin{aligned} d(k, B(\gamma)(\omega)) &= d(k, S^2_{C(\omega,\cdot)}) = \inf\{\|k-g\|_2\colon g\in S^2_{C(\omega,\cdot)}\} \\ &= \inf\Big\{\int_0^b \|k(s)-g(s)\|\,ds\colon\ g\in S^2_{C(\omega,\cdot)}\Big\} = \int_0^b \inf\{\|k(s)-z\|\,ds\colon z\in C(\omega, s)\}\,ds \\ &= \int_0^b d(k(s),\ C(\omega, s))\,ds. \end{aligned}$$

Thus from Fubini's theorem we see that $\omega\to d(k, B(\gamma)(\omega))$ is measurable, and hence $\omega\to B(\gamma)(\omega)$ is measurable.

Let $r\colon \Omega\times L^2(X)\to C(T, X)$ be the map that to each $(\omega, h)\in\Omega\times L^2(X)$ assigns the unique strong solution of

$$-\dot{x}(t)\in\partial\varphi(\omega, x(t))+h(t) \text{ a.e.,} \qquad x(0)=x_0(\omega).$$

Fix $h\in L^2(X)$. From Lemma 2.1 of [2] we know that

$$r(\omega, h)=\lim_{\lambda\to 0} r_\lambda(\omega, h),$$

where $r_\lambda(\omega, h)$ is the unique solution of

$$\dot{x}_\lambda(t)=\nabla\varphi_\lambda(\omega, x_\lambda(t))+h(t), \qquad x_\lambda(0)=x_0(\omega)$$

with

$$\varphi_\lambda(\omega, x)=\inf\left[\frac{1}{2\lambda}\|x-z\|^2+\varphi(\omega, z)\colon z\in X\right].$$

But $r_\lambda(\cdot, h)$ is measurable, since $\nabla\varphi_\lambda(\omega, x)$ is measurable in $\omega$ (see Theorem 2.3 of [1]). Next fix $\omega\in\Omega$ and consider the map $h\to r(\omega, h)$. We claim that it is continuous from $(L^2(X), \mathrm{w})$ into $C(T, X)$. So let $h_n\xrightarrow{w} h$ in $L^2(X)$. Set $x_n=r(\omega, h_n)$. Let $t, t'\in T$, $t<t'$. We have, for all $n\geqslant 1$,

$$\begin{aligned}\|x_n(t')-x_n(t)\| &= \left\|\int_t^{t'}\dot{x}_n(s)ds\right\|\leqslant\int_t^{t'}\|\dot{x}_n(s)\|\,ds\\ &\leqslant\int_0^b\|\chi_{[t,t']}(s)\dot{x}_n(s)\|\,ds\leqslant\left[\int_0^b\|\chi_{[t,t']}(s)\|^2ds\right]^{1/2}\left[\int_0^b\|\dot{x}_n(s)\|^2ds\right]^{1/2}.\end{aligned}$$

From the estimates in Theorem 3.6 of [6] we know that

$$\int_0^b\|\dot{x}_n(s)\|^2ds\leqslant\sup_{n\geqslant 1}\|h_n\|_2+[\varphi(\omega, x_0(\omega))]^{1/2}=M_1(\omega)$$
$$\Rightarrow\ \|x_n(t')-x_n(t)\|\leqslant M_1(\omega)(t'-t)^{1/2},$$

which implies that $V=\{x_n\}_{n\geqslant 1}$ is equicontinuous.

Also again from Theorem 3.6 of [6] we know that

$$\|\dot{x}_n(t)\|^2+\frac{d}{dt}\varphi(\omega, x_n(t))=(h_n(t), \dot{x}_n(t)) \text{ a.e.,}$$

so

$$\begin{aligned}\varphi(\omega, x_n(t)) &\leqslant\varphi(\omega, x_0(\omega))+\int_0^t(h_n(s), \dot{x}_n(s))ds\\ &\leqslant\varphi(\omega, x_0(\omega))+[M_1(\omega)]^{1/2}\sup_{n\geqslant 1}\|h_n\|_2=M_2(\omega) \qquad \text{for all } n\geqslant 1.\end{aligned}$$

Recalling that $\varphi(\omega, \cdot)$ is inf-compact, we deduce that $V(t)=\mathrm{cl}\{x_n(t)\}_{n\geqslant 1}$ is compact. Invoking the Arzelà–Ascoli theorem we conclude that $V$ is relatively compact in $C(T, X)$. Hence by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, X)$. By definition we have

$$-\dot{x}_n(t) \in \partial\varphi(\omega, x_n(t)) + h_n(t) \text{ a.e.,} \qquad h_n \in S^2_{\hat{F}(\omega,\cdot,x_n(\cdot))},$$

so

$$-\dot{x}_n(t) - h_n(t) \in \partial\varphi(\omega, x_n(t)) \text{ a.e.,}$$

and hence

$$(-\dot{x}_n - h_n)(\cdot) \in S^2_{\partial\varphi(\omega,x_n(\cdot))}.$$

But if

$$I(\varphi_n(\omega))(x_n) = \int_0^b \varphi(\omega, x_n(t))dt,$$

from [15] (Theorem 22) we know that

$$S^2_{\partial\varphi(\omega,\cdot,x_n(\cdot))} = \partial I(\varphi(\omega))(x_n).$$

Thus we have

$$(-\dot{x}_n - h_n)(\cdot) \in \partial I(\varphi(\omega))(x_n).$$

By hypothesis, $h_n \xrightarrow{w} h$ in $L^2(X)$, while since $L^2(X)$ is a Hilbert space and $\{\dot{x}_n\}_{n\geq 1}$ is bounded, by passing to a subsequence if necessary we may assume that $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(X)$. So we have

$$(x_n, -\dot{x}_n - h_n) \xrightarrow{s \times w} (x, -\dot{x} - h).$$

Since $\partial I(\varphi(\omega))(\cdot)$ is maximal monotone, its graph is demiclosed. Therefore we see that

$$(x, -\dot{x} - h) \in \mathrm{Gr}\, \partial I(\varphi(\omega))(\cdot) \Rightarrow (-\dot{x} - h)(\cdot) \in \partial I(\varphi(\omega))(x) = S^2_{\partial\varphi(\omega,x(\cdot))}$$
$$\Rightarrow -\dot{x}(t) \in \partial\varphi(\omega, x(t)) + h(t) \text{ a.e. } \Rightarrow x = r(\omega, h),$$

which implies that $h \to r(\omega, h)$ is continuous.

Now consider the multifunction $R$: $\Omega \times L^2(X) \to 2^{L^2(X)} \backslash \{\emptyset\}$ defined by

$$R(\omega, h) = S^2_{\hat{F}(\omega,\cdot,r(\omega,h)(\cdot))}.$$

Let $u(\omega, t, h) = e_t(r(\omega, h))$, where $e_t(\cdot)$ is the evaluation map at $t \in T$. It is well known that $(t, h) \to u(\omega, t, h)$ is continuous, while clearly $\omega \to u(\omega, t, h)$ is measurable. Thus $(\omega, t, h) \to u(\omega, t, h)$ is measurable. Let

$$\psi_1\colon (\omega, t, h) \to (\omega, t, u(\omega, t, h)).$$

This is measurable. Also let $y \in X$ and let

$$\psi_2\colon (\omega, t, h) \to d(y, \hat{F}(\omega, t, x)).$$

Because of hypothesis $H(F)$ (i) we deduce that $\psi_2(\cdot,\cdot,\cdot)$ is measurable. So finally we infer that

$$(\omega, t, h) \to (\psi_2 \circ \psi_1)(\omega, t, h) = d(x, \hat{F}(\omega, t, r(\omega, h)(t)))$$

is measurable, while clearly

$$x \to d(x, \hat{F}(\omega, t, r(\omega, h)(t)))$$

is continuous. Therefore we can say that

$$(\omega, t, h, x) \to d(x, \hat{F}(\omega, t, r(\omega, h)(t)))$$

is measurable, and so

$$(\omega, t, h) \to d(g(t), \hat{F}(\omega, t, r(\omega, h)(t)))$$

is measurable and, consequently,

$$(\omega, h) \to \int_0^b d(g(t), \hat{F}(\omega, t, r(\omega, h)(t)))dt = d(g, S^2_{\hat{F}(\omega,\cdot,r(\omega,h)(\cdot))})$$

is measurable, while is continuous in $g$. Hence

$$k\colon (\omega, h, g) \to d(g, S^2_{\hat{F}(\omega,\cdot,r(\omega,h)(\cdot))})$$

is measurable, and so

$$\begin{aligned} \mathrm{Gr}R = \{(\omega, h, g) \in \Omega \times L^2(X) \times L^2(X)\colon k(\omega, h, g) = 0\} \\ \in \Sigma \times B(L^2(X) \times L^2(X)) = \Sigma \times B(L^2(X)) \times B(L^2(X)). \end{aligned}$$

Next we claim that, for fixed $\omega \in \Omega$, $R(\omega, \cdot)$ is u.s.c. on $B(\gamma)(\omega)$ with the weak $L^2(X)$-topology. Note that

$$R(\omega, \cdot)\colon B(\gamma)(\omega) \to P_{fc}(B(\gamma)(\omega)).$$

Since $B(\gamma)(\omega)$ is w-compact in $L^2(X)$ (being bounded), it suffices to show that $\mathrm{Gr}R(\omega, \cdot)$ is closed in $B(\gamma)(\omega) \times B(\gamma)(\omega)$ with the product weak topology. Since the latter is metrizable, it suffices to consider sequences. So let

$$(h_n, g_n) \in \mathrm{Gr}R(\omega, \cdot), \quad (h_n, g_n) \xrightarrow{w \times w} (h, g) \text{ in } L^2(X) \times L^2(X).$$

Then we have

$$r(\omega, h_n) \to r(\omega, h) \text{ in } C(T, X),$$

while from Theorem 3.1 of [14] we get

$$\begin{aligned} & g(t) \in \overline{\mathrm{conv}}\ \overline{\text{w-lim}}\{g(t)\}_{n \geq 1} \text{ a.e.} \\ & \subseteq \overline{\mathrm{conv}}\, \overline{\text{w-lim}}\, \hat{F}(\omega, t, r(\omega, h_n)(t)) \\ & \subseteq \overline{\mathrm{conv}}\, \hat{F}(\omega, t, r(\omega, h)(t)) = \hat{F}(\omega, t, r(\omega, h)(t)) \text{ a.e.} \Rightarrow g \in S^2_{\hat{F}(\omega,\cdot,r(\omega,h)(\cdot))} \Rightarrow g \in R(\omega, h), \end{aligned}$$

and so $\mathrm{Gr}R(\omega,\cdot)$ is closed, which implies that $R(\omega,\cdot)$ is u.s.c. from $B(\gamma)(\omega)$ into itself.

Let $L(\omega)=\{f\in B(\gamma)(\omega)\colon f\in R(\omega,f)\}$. From the Kakutani–Ky Fan fixed point theorem we know that, for all $\omega\in\Omega$, $L(\omega)\neq\emptyset$. Also

$$\begin{aligned}\mathrm{Gr}L&=\{(\omega,f)\in\Omega\times L^2(X)\colon f\in R(\omega,f)\}\cap\mathrm{Gr}(B(\gamma)(\cdot))\\&=\mathrm{proj}_{\Omega\times L^2(X)}[(\Omega\times D)\cap\mathrm{Gr}R]\cap\mathrm{Gr}(B(\gamma)(\cdot)),\end{aligned}$$

where

$$D=\{(f_1,f_2)\in L^2(X)\times L^2(X)\colon f_1=f_2\}.$$

We saw earlier in the proof that $\mathrm{Gr}R\in\Sigma\times B(L^2(X))\times B(L^2(X))$. Thus

$$(\Omega\times D)\cap\mathrm{Gr}R\in\Sigma\times B(L^2(X))\times B(L^2(X)).$$

Using Theorem 39.IV.1 of [12] we get

$$\mathrm{proj}_{\Omega\times L^2(X)}[(\Omega\times D)\cap\mathrm{Gr}R]\in\Sigma\times B(L^2(X)).$$

Recalling that $\mathrm{Gr}B(\gamma)(\cdot)\in\Sigma\times B(L^2(X))$, we conclude that

$$\mathrm{Gr}L\in\Sigma\times B(L^2(X)).$$

Apply Theorem 3 of [16] to get $v\colon\Omega\to L^2(X)$ measurable such that $v(\omega)\in L(\omega)$ for all $\omega\in\Omega$. Then let

$$x(\omega,t)=r(\omega,v(\omega))(t).$$

Clearly, this is a random solution of (*) with perturbation $\hat{F}(\omega,t,x)$. Since

$$|\hat{F}(\omega,t,x)|\leqslant\alpha(\omega,t)+\|x\|\,b(\omega,t)$$

(recall its definition), with a similar argument as in the beginning of the proof, we see that $\|x(\omega,t)\|\leqslant M(\omega)$, and so

$$\hat{F}(\omega,t,x(\omega,t))=F(\omega,t,x(\omega,t)),$$

which implies that $x(\cdot,\cdot)$ is the desired random solution of (*).

**3. Examples.**

1. Let $G$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial G=\Gamma$. Consider the following random, nonlinear, multivalued diffusion equation defined on $T=[0,b]\times G$:

$$(**)_1\quad\begin{aligned}&-\frac{\partial v(\omega,t,z)}{\partial t}+\Delta v(\omega,t,z)-\partial k(\omega,v(\omega,t,z))\in F(\omega,t,z,v(\omega,t,z)),\\&v(\omega,0,z)=v_0(\omega,z)\ \text{on}\ \Omega\times\{0\}\times G,\qquad v(\omega,t,z)=0\ \text{on}\ \Omega\times T\times\Gamma.\end{aligned}$$

Here $F\colon\Omega\times T\times\mathbf{R}^n\times\mathbf{R}\to P_{fc}(\mathbf{R})$ is a multifunction such that

(i) $(\omega, t, z, q) \to F(\omega, t, z, q)$ is measurable, $w \to d\big(g(\cdot), F(\omega, t, \cdot, w(\cdot))\big)$ is continuous on $L^2(G)$ for all $g \in L^2(G)$.

(ii) for all $(\omega, t, z) \in \Omega \times T \times \boldsymbol{R}^n$, $q \to F(\omega, t, z, q)$ is closed;

(iii) $|F(\omega, t, z, q)| \leqslant \alpha(\omega, t, z) + b(\omega, t, z)|q|$, where $\alpha(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ are measurable, and $\alpha(\omega, \cdot, \cdot)$, $b(\omega, \cdot, \cdot) \in L^2(T \times \boldsymbol{R}^n)$.

Also, let $k$: $\Omega \times \boldsymbol{R} \to \bar{\boldsymbol{R}}$ be a convex normal integrand such that

$$\partial k(\omega, 0) \neq \emptyset.$$

Define $\varphi$: $\Omega \times L^2(G) \to \bar{\boldsymbol{R}}$ by

$$\varphi(\omega, x) = \begin{cases} \frac{1}{2}\int_G |\nabla x|^2 dz + \int_\Omega k(\omega, x(z))dz & \text{for } x \in H_0^1(G), k(\omega, x(\cdot)) \in L^1(G), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, $(\omega, x) \to \varphi(\omega, x)$ is measurable, and from [2] (see the remark in page 88) we know that, for all $\omega \in \Omega$, $\varphi(\omega, \cdot)$ is l.s.c. on $L^2(G)$. So $\varphi(\cdot,\cdot)$ is a convex normal integrand. We also have

$$\partial\varphi(\omega, x) = -\Delta x + \partial k(\omega, x)$$

and

$$D(\partial\varphi(\omega, \cdot)) = \{x \in H_0^1(G) \cap H^2(G) \cap \hat{D}\text{: there exists } y \in L^2(G)$$
$$\text{such that } y(z) \in \partial k(\omega, x(z)) - \Delta x(z) \text{ a.e.}\},$$

where

$$\hat{D} = \{x \in L^2(G)\text{: } S^2_{\partial k(\omega, x(\cdot))} \neq \emptyset\}.$$

Recalling that $\omega \to \partial k(\omega, v)$ is measurable, we can easily check that $D(\omega) = D(\partial\varphi(\omega, \cdot))$ has a graph in $\Sigma \times B(L^2(G))$, and so applying Theorem 3 of [16] we deduce that $D(\cdot)$ admits a measurable selector. Note also that for fixed $\omega \in \Omega$ we have

$$\{x \in L^2(G)\text{: } \varphi(\omega, x) \leqslant \lambda\} = \{x \in L^2(G)\text{: } \tfrac{1}{2}\int_G |\nabla x|^2 dz + \int_G k(\omega, x(z))dz \leqslant \lambda\}$$
$$\subseteq \{x \in L^2(G)\text{: } \tfrac{1}{2}\int_G |\nabla x|^2 dz \leqslant \lambda\},$$

and by Poincaré's inequality we deduce that this last set is bounded in $H_0^1(G)$. But the imbedding of $H_0^1(G)$ into $L^2(G)$ is compact (Rellich's theorem). So $\varphi(\omega, \cdot)$ is inf-compact in $X = L^2(G)$.

Also, let $\hat{F}$: $\Omega \times T \times L^2(G) \to P_{fc}(L^2(G))$ be defined by

$$\hat{F}(\omega, t, w) = S^2_{F(\omega, t, \cdot, w(\cdot))}.$$

For $g \in L^2(G)$ we have

$$d(g, \hat{F}(\omega, t, w)) = \int_G d\big(g(z), F(\omega, t, z, w(z))\big)dz.$$

Because of hypothesis (i) about $F$, we conclude that

$$(\omega, t, w) \to d(g, \hat{F}(\omega, t, w))$$

is measurable, so $(\omega, t, w) \to \hat{F}(\omega, t, w)$ is measurable. Also, if

$$(w_n, v_n) \in \operatorname{Gr}\hat{F}(\omega, t, \cdot) \quad \text{and} \quad (w_n, v_n) \xrightarrow{s \times w} (w, v) \text{ in } L^2(G) \times L^2(G),$$

then using Theorems 3.1 and 4.2 of [14] we get

$$v(z) \in \overline{\operatorname{conv}}\, \text{w-}\overline{\lim}\{v_n(z)\}_{n \geq 1} \text{ a.e.} \subseteq \overline{\operatorname{conv}}\, \overline{\lim} F(\omega, t, z, w_n(z))$$
$$\subseteq F(\omega, t, z, w(z)) \text{ a.e. on } G,$$

which implies that $\hat{F}(\omega, t, \cdot)$ is sequentially closed in $X \times X_w$, $X = L^2(G)$.

Finally, observe that

$$|\hat{F}(\omega, t, w)| \leq \hat{a}(\omega, t) + \|w\|_2 \hat{b}(\omega, t),$$

where

$$\hat{a}(\omega, t) = \|\alpha(\omega, t, \cdot)\|_2 \lambda(G)^{1/2} \quad \text{and} \quad \hat{b}(\omega, t) = \|b(\omega, t, \cdot)\|_2.$$

Now rewrite $(**)_1$ as the following abstract evolution inclusion:

$$(**)_1' \qquad -\dot{x}(\omega, t) \in \partial\varphi(\omega, x(\omega, t)) + \hat{F}(\omega, t, x(\omega, t)), \qquad x(\omega, 0) = x_0(\omega),$$

where $x_0 \colon \Omega \to L^2(G)$ is defined by $x_0(\omega)(\cdot) = v_0(\omega, \cdot)$. From the above we see that hypotheses $H(\varphi)$, $H(\hat{F})$ and $H_0$ are all satisfied. So we can apply the Theorem and get a random solution $x \colon \Omega \times T \to L^2(G)$. Set $v(\omega, t, z) = x(\omega, t)(z)$. This is the desired random generalized solution of $(**)$.

2. Let $G$ be as in the previous example. Consider the following boundary-initial value problem on $\Omega \times T \times G$:

$$(**)_2 \quad \begin{aligned} &\frac{\partial v}{\partial t} - \sum_{k=1}^{n} c_k(\omega) \frac{\partial}{\partial z_k}\left(\left|\frac{\partial v}{\partial z_k}\right|^{r-2} \frac{\partial v}{\partial z_k}\right) \in F(\omega, t, z, v(\omega, t, z)), \\ &v(\omega, t, z) = 0 \text{ on } \Omega \times T \times \Gamma, \qquad v(\omega, 0, z) = v_0(\omega, z) \text{ on } \Omega \times G. \end{aligned}$$

Here $F(\omega, t, z, q)$ is as in Example 1, while $c_k \colon \Omega \to \mathbf{R}$, $k \in \{1, \ldots, n\}$, are measurable functions.

Let $A(\omega) \colon W_0^{1,p}(G) \to W^{-1,q}(G)$, $1 < p, q < \infty$, $1/p + 1/q = 1$, be defined by the following Dirichlet form:

$$\alpha(\omega, w, v) = (A(\omega)w, v) = \sum_{k=1}^{n} c(\omega) \int_G \left|\frac{dw}{dz_k}\right|^{r-2} \frac{dw}{dz_k}\frac{dv}{dz_k} dz, \qquad v, w \in W_0^{1,p}(\Omega).$$

Note that if $\varphi \colon \Omega \times W_0^{1,p}(G) \to \bar{\mathbf{R}}$ is

$$\varphi(\omega, v) = \frac{1}{r} \sum_{k=1}^{n} c(\omega) \int_G \left|\frac{\partial v}{\partial z_k}\right|^r dz,$$

then $A(\omega)v = \nabla\varphi(\omega, v)$. Also, let

$$D(\omega) = \{v \in W_0^{1,p}(G) \colon A(\omega)v \in L^2(G)\}.$$

Then $A(\omega)|_{D(\omega)}$ is equal to $\partial\psi(\omega, \cdot)$, where

$$\psi(\omega, v) = \begin{cases} \varphi(\omega, v) & \text{if } v \in W_0^{2,p}(G), \\ +\infty & \text{otherwise.} \end{cases}$$

Rewrite $(**)_2$ as

$$(**)_2' \qquad -\dot{x}(\omega, t) \in \partial\psi(\omega, x(\omega, t)) + \hat{F}(\omega, t, x(\omega, t)), \qquad x(\omega, 0) = x_0(\omega).$$

Taking $X = L^2(G)$, we can check as before that all hypotheses of the Theorem are satisfied. So there exists a random solution for $(**)_2'$, which gives us the random generalized solution of $(**)_2$.

3. Let $G$ be as before and let $r > (n-2)/n$. Consider the following initial-boundary value problem defined on $\Omega \times G$:

$$(**)_3 \quad \begin{aligned} &-\frac{\partial v(\omega, t, z)}{\partial t} + \Delta v(\omega, t, z)|v(\omega, t, z)|^{r-1} \in F(\omega, t, z, v(\omega, t, z)), \\ &v(\omega, t, z) = 0 \text{ on } \Omega \times T \times \Gamma, \quad v(\omega, 0, z) = v_0(\omega, z) \text{ on } \Omega \times \{0\} \times G. \end{aligned}$$

Take $X = L^2(G)$ and $Ax = -\Delta x|x|^{r-1}$ on

$$D(A) = \{x \in L^2(G)\colon x,\ x^{r-1} \in H_0^1(G) \text{ and } \Delta x|x|^{r-1} \in L^2(G)\}.$$

Then it is well known (see [2], p. 67) that $Ax = \partial\varphi(x)$, where $\varphi\colon H^{-1}(G) \to \bar{R}$ is given by

$$\varphi(x) = \begin{cases} \int_G |x(z)|^r dz & \text{if } x(\cdot) \in L^r(G), \\ +\infty & \text{otherwise.} \end{cases}$$

So $(**)_3$ takes the form

$$(**)_3' \qquad -\dot{x}(\omega, t) \in \partial\varphi(x(\omega, t)) + \hat{F}(\omega, t, x(\omega, t)), \qquad x(\omega, 0) = x_0(\omega).$$

On $(**)_3'$ we apply the Theorem to conclude the existence of a random solution. Such problems appear in mathematical physics and random free boundary problems. Also, when

$$F(\omega, t, z, q) = [\underline{\lim}_{q' \to q} f(\omega, t, z, q'),\ \overline{\lim}_{q' \to q} f(\omega, t, z, q')]$$

and the two limits are both measurable and l.s.c. and u.s.c., respectively, we get random evolution equations that describe obstacle problems (see [7]).

4. For this example let $K\colon \Omega \to P_{kc}(X)$, where $P_{kc}(X)$ stands for a family of nonempty, compact, convex subsets of the separable Hilbert space $X$, and set

$$\varphi(\omega, x) = \partial\delta_{K(\omega)}(x),$$

where $\delta_{K(\omega)}(x) = 0$ if $x \in K(\omega)$, and $\delta_{K(\omega)}(x) = +\infty$ otherwise (the indicator function of the set $K(\cdot)$). Then, recalling that $\partial\delta_{K(\omega)}(x) = N_{K(\omega)}(x)$ is a normal cone to $K(\omega)$ at $x$, equation $(*)$ takes the form

$$(**)_4 \qquad -\dot{x}(\omega, t) \in N_{K(\omega)}(x(\omega, t)) + F(\omega, t, x(\omega, t)), \qquad x(\omega, 0) = x_0(\omega).$$

This way we obtain a random version of the "sweeping process" problem of Moreau [13] that has applications in mechanics.

**Acknowledgments.** This work was done while the second-named author was visiting the University of Thessaloniki, School of Technology, Mathematics Division, Thessaloniki 540-06, Greece. Financial support was provided by N.S.F. Grant DMS 8602313.

*REFERENCES*

[1] H. Attouch, *Familles d'opérateurs maximaux monotones et mesurabilité*, Ann. Mat. Pura Appl. 120 (1979), pp. 35–111.

[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[3] G. Becus, *Random generalized solutions to the heat equation*, J. Math. Anal. Appl. 60 (1977), pp. 93–102.

[4] — *Solutions to the random heat equation by the method of successive approximations*, ibidem 64 (1978), pp. 277–296.

[5] A. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York 1972.

[6] H. Brezis, *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, Math. Studies, Vol. 5, North-Holland, Amsterdam 1973.

[7] K-C. Chang, *Free boundary problems and the set valued mappings*, J. Differential Equations 49 (1983), pp. 1–28.

[8] P. L. Chow, *Applications of function space integrals to problems in wave propagation in random media*, J. Math. Phys. 13 (1972), pp. 1224–1236.

[9] J. Delahaye and J. Denel, *The continuities of the point-to-set maps, definitions and equivalences*, Math. Programming Study 10 (1979), pp. 8–12.

[10] J. Kampé de Fériet, *Intégrales aléatoires de l'équation de la chaleur dans une barre infinie*, C. R. Acad. Sci. Paris 240 (1955), pp. 710–712.

[11] — *Random solutions of partial differential equations*, pp. 199–208 in: Proc. 3rd Berkeley Symp. Math. Stat. and Probability 1955, Vol. III, Univ. of California Press, Berkeley, California, 1959.

[12] K. Kuratowski, *Topology. I*, Academic Press, New York 1966.

[13] J.-J. Moreau, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations 26 (1977), pp. 347–374.

[14] N. S. Papageorgiou, *Convergence theorems for Banach space valued integrable multifunctions*, Internat. J. Math. Math. Sci. 10 (1987), pp. 433–442.

[15] R. T. Rockafellar, *Conjugate duality and optimization*, Conference Board of Mathematical Sciences Series, No. 16, SIAM Publications, Philadelphia 1974.

[16] M.-F. Sainte-Beuve, *On the extension of von Neumann–Aumann's theorem*, J. Funct. Anal. 17 (1974), pp. 112–129.

[17] T. C. Samuels, *Heat conduction in solids with random external temperatures and/or random internal heat generation*, Internat. J. Heat Mass Transfer 9 (1966), pp. 301–314.

[18] T. T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York 1973.

[19] C. P. Tsokos and W. J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York 1974.

NATIONAL TECHNICAL UNIVERSITY OF ATHENS
ATHENS, GREECE

1015 DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616, U.S.A.

*Reçu par la Rédaction le 10.5.1988*