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FIXED POINTS OF HOLOMORPHIC MAPPINGS IN THE HILBERT BALL

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A recent paper of Goebel et al. [2], in which the authors have studied metrical convexity in the unit ball in a Hilbert space with a hyperbolic metric and applied it to the theory of fixed points of holomorphic mappings, was a motivation for our remarks placed in this paper. The results of theirs and ours are closely related to the earlier work of Earle and Hamilton [1].

Let H be a complex Hilbert space and B the open unit ball in H. In B we have the so-called hyperbolic metric (see [3])

$$\varrho(x, y) = \tanh^{-1}(1-\sigma(x, y))^{1/2},$$

where

$$\sigma(x, y) = \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-(x, y)|^2}.$$

This metric has nice properties (see [2]) and here we will show some more new ones.

It is known [6] that in (B, ϱ) every ball is an ellipsoid

$$\left\|y_1 - \frac{k}{1+k\|x\|^2}x\right\|^2 + \frac{\|y_2\|^2}{1+k\|x\|^2} = \frac{k(\|x\|^2-1)+1}{(1+k\|x\|^2)^2}$$

if $y \in \partial K(x, r)$, $y = y_1 + y_2$ (y_1 is the orthogonal projection of y on the span $\{x\}$) and

$$k = \frac{1+\tanh^2 r}{1-\|x\|^2}.$$

If the dimension of H is not less than 2, then all balls centered at a non-zero point in (B, ϱ) are not balls in H. Next all these balls are uniformly convex in a usual sense.

LEMMA 1. *A modulus of convexity*

$$\delta(x, r, \varepsilon, t) = 1-\frac{1}{r}\sup\{\varrho(x, ty+(1-t)z)\colon \varrho(x, y)\leqslant r,\ \varrho(x, z)\leqslant r,\ \varrho(y, z)\geqslant \varepsilon r\}$$

($x\in B$, $0<r<\infty$, $0<\varepsilon<2$, $0<t<1$) *has the following properties*:

(a) *If* $\lim x_n = x$, $\lim r_n = r$, $\lim \varepsilon_n = \varepsilon$, *and* $\lim t_n = t$, *then*

$$\delta(x, r, \varepsilon, t)\leqslant \liminf \delta(x_n, r_n, \varepsilon_n, t_n).$$

(b) *If the conditions*

1° $\{x_n\}$ *is weakly convergent to* x *and* $\lim\|x_n\|$ *exists,*

2° *for* $y\neq z$, $y, z\in B$, *we have*

$$\varrho(x_n, y)\leqslant r_n,\quad \varrho(x_n, z)\leqslant r_n\quad (n = 1, 2, \ldots)$$

and

$$\lim r_n = r,$$

3° e *is orthogonal to* x, y, z *and*

$$\|x+e\| = \lim\|x_n\|$$

are fulfilled, then

$$\lim \varrho\left(x_n, \frac{y+z}{2}\right)\leqslant\left[1-\delta\left(x+e, r, \frac{\varrho(y, z)}{r}, \frac{1}{2}\right)\right]r.$$

Proof. Since we may restrict our considerations to the three-dimensional Hilbert space H, the property (a) of our lemma is true. In (b) we have

$$\varrho(x+e, y)\leqslant r,\quad \varrho(x+e, z)\leqslant r$$

and

$$\lim \varrho\left(x_n, \frac{y+z}{2}\right) = \varrho\left(x+e, \frac{y+z}{2}\right)\leqslant\left[1-\delta\left(x+e, r, \frac{\varrho(y, z)}{r}, \frac{1}{2}\right)\right]r.$$

COROLLARY 1. *Let* X *be a non-empty subset of* B, *closed in* (B, ϱ) *and convex in* H. *Then for any* $x\in B$ *there exists exactly one point* $y\in X$ *such that* $\varrho(x, y) = \operatorname{dist}(x, X)$. *This metrical projection is continuous.*

Proof. The sequence of sets

$$\{X_n\} = \{\{z\in X\colon \varrho(x, z)\leqslant \operatorname{dist}(x, X)+1/n\}\}$$

consists of non-empty, bounded, closed and convex subsets of H, and hence

$$\bigcap_n X_n\neq\emptyset.$$

From (a) of Lemma 1 we infer that $\lim \operatorname{diam} X_n$ is equal to 0 and that this projection is continuous.

Now we notice the following useful property of σ:

LEMMA 2. *If sequences $\{x_n\}$ and $\{y_n\}$ of elements of B are weakly convergent to $x \in B$, then for any $y \in B$ we have*

$$\mathrm{LIM}\frac{\sigma(y_n, y)}{\sigma(x_n, x)} = \sigma(x, y)\,\mathrm{LIM}\frac{1-\|y_n\|^2}{1-\|x_n\|^2},$$

where LIM *denotes one of the following limits (the same on both sides of the equality):*

$$\liminf\frac{\sigma(y_n, y)}{\sigma(x_n, x)}, \quad \limsup\frac{\sigma(y_n, y)}{\sigma(x_n, x)},$$
$$\frac{\liminf\sigma(y_n, y)}{\liminf\sigma(x_n, x)}, \quad \frac{\liminf\sigma(y_n, y)}{\limsup\sigma(x_n, x)}, \quad \ldots,$$

whenever it makes sense.

Proof. Observe that

$$\mathrm{LIM}\frac{\sigma(y_n, y)}{\sigma(x_n, x)} = \mathrm{LIM}\frac{\dfrac{(1-\|y_n\|^2)(1-\|y\|^2)}{|1-(y_n, y)|^2}}{\dfrac{(1-\|x_n\|^2)(1-\|x\|^2)}{|1-(x_n, x)|^2}}$$
$$= \mathrm{LIM}\frac{\dfrac{(1-\|y_n\|^2)(1-\|y\|^2)}{|1-(x, y)|^2}}{\dfrac{(1-\|x_n\|^2)(1-\|x\|^2)}{(1-\|x\|^2)^2}}$$
$$= \sigma(x, y)\,\mathrm{LIM}\frac{1-\|y_n\|^2}{1-\|x_n\|^2}.$$

This lemma implies a few corollaries.

COROLLARY 2 (see the proof of Theorem 15 in [2]). *Under the assumptions of Lemma* 1 *and $\|y_n\| \geqslant \|x_n\|$ for $n = 1, 2, \ldots$ the following inequality is true:*

$$\mathrm{LIM}\frac{\sigma(y_n, y)}{\sigma(x_n, x)} \leqslant \sigma(x, y).$$

COROLLARY 3 (see Theorem 7 in [2]). *If $\{x_n\}$ is a ϱ-bounded sequence which converges weakly to x, and y is an element of B, then*

$$\liminf\sigma(x_n, y) = \sigma(x, y)\liminf\sigma(x_n, x).$$

As Goebel et al. [1] showed, an asymptotic center of a ϱ-bounded sequence is also a very useful tool in investigations of fixed points of holomorphic mappings. We will give a few remarks on this matter.

Let X be a non-empty subset of B. We choose an arbitrary ϱ-bounded sequence $\{x_n\}$ and a point x in B. The number

$$r(x, \{x_n\}) = \limsup \varrho(x_n, x)$$

is called an *asymptotic radius of* $\{x_n\}$ *at* x, and the number

$$r_X(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\})$$

is an *asymptotic radius of* $\{x_n\}$ *with respect to* X (or *in* X). The set

$$A(X, \{x_n\}) = \{x\in X\colon\ r(x, \{x_n\}) = r_X(\{x_n\})\}$$

is called an *asymptotic center* of $\{x_n\}$ in X.

In [2] it was proved that $A(B, \{x_n\})$ contains only one point. Here we will show some more.

THEOREM 1. *If X is non-empty, closed in (B, ϱ) and convex in H, then any ϱ-bounded sequence in B has asymptotic center in X containing only one point.*

Proof. Take $\varepsilon > 0$ and put

$$A(X, \{x_n\}, \varepsilon) = \{x\in X\colon\ r(x, \{x_n\}) \leqslant r_X(\{x_n\})+\varepsilon\}.$$

Notice that every $A(X, \{x_n\}, \varepsilon)$ is a non-empty, bounded, closed and convex subset of H. Therefore

$$A(X, \{x_n\}) = \bigcap_{\varepsilon>0} A(X, \{x_n\}, \varepsilon)$$

is non-empty and convex. To prove the uniqueness suppose that $y, z\in A(X, \{x_n\})$ and $y \neq z$. Thus we have

$$\limsup \varrho(x_n, y) = r_X(\{x_n\}),$$

$$\limsup \varrho(x_n, z) = r_X(\{x_n\}),$$

$$\limsup \varrho\left(x_n, \frac{y+z}{2}\right) = \lim \varrho\left(x_{n_i}, \frac{y+z}{2}\right) = r_X(\{x_n\}).$$

Hence for $x_{n_i} \rightharpoonup x$ with $\lim \|x_{n_i}\|$ (see (b) of Lemma 1)

$$0 < r_X(\{x_n\}) = \lim \varrho\left(x_{n_i}, \frac{y+z}{2}\right)$$

$$\leqslant \left[1-\delta\left(x+e, r_X(\{x_n\}), \frac{\varrho(y, z)}{r_X(\{x_n\})}, \frac{1}{2}\right)\right] r_X(\{x_n\}),$$

where e is chosen in a similar way as in (b) of Lemma 1. We get a contradiction.

Corollary 3 states that the weak limit of a weakly convergent ϱ-bounded sequence coincides with its asymptotic center in B. The next theorem will show that the asymptotic center of any ϱ-bounded sequence in B lies in the ϱ-convex closed hull of weak limits of its subsequences (see also Theorem 6 in [2]).

THEOREM 2. *For every ϱ-bounded sequence $\{x_n\}$ we have*

$$A(B, \{x_n\}) \in A = \varrho\text{-conv}\,\{x \in B\colon \bigvee_{\{x_{n_i}\}} x_{n_i} \rightharpoonup x\},$$

where ϱ-conv *denotes a ϱ-convex closed hull.*

Proof. Let us take $y \notin B$ and let z be a metric projection of the element y on the set A (see Theorem 3 in [2]). Then there exists a subsequence $\{x_{n_i}\}$ such that

$$\liminf \sigma(x_n, z) = \lim \sigma(x_{n_i}, z)$$

and $\{x_{n_i}\}$ is weakly convergent to $x \in A$. Moreover, we have $\sigma(x, z) > \sigma(x, y)$ (see [2]). Thus

$$\begin{aligned}\liminf \sigma(x_n, z) &= \lim \sigma(x_{n_i}, z) = \lim \frac{(1-\|x_{n_i}\|^2)(1-\|z\|^2)}{|1-(x_{n_i}, z)|^2}\\ &= \frac{1-\|z\|^2}{|1-(x, z)|^2}(1-\lim \|x_{n_i}\|^2) = \sigma(x, z)\frac{1-\lim\|x_{n_i}\|^2}{1-\|x\|^2}\\ &> \sigma(x, y)\frac{1-\lim\|x_{n_i}\|^2}{1-\|x\|^2} = \frac{1-\|y\|^2}{|1-(x, y)|^2}(1-\lim\|x_{n_i}\|^2)\\ &= \lim \sigma(x_{n_i}, y) \geqslant \liminf \sigma(x_n, y),\end{aligned}$$

and therefore y cannot belong to the asymptotic center of $\{x_n\}$.

Now we are concerned with holomorphic mappings. It is known that each holomorphic mapping $T\colon B \to B$ is non-expansive in (B, ϱ), i.e.,

$$\varrho(Tx, Ty) \leqslant \varrho(x, y) \quad (x, y \in B),$$

and for any points $x, y \in B$ there exists a biholomorphic mapping which maps x on y and this mapping is obviously a ϱ-isometry.

A subset $X \subset B$ is said to be *ϱ-starshaped* if there exists $x \in X$ such that for every $y \in X$ the ϱ-segment joining x with y lies in X.

THEOREM 3. *Suppose $T\colon B \to B$ is holomorphic. Then T has a fixed point iff there exists a ϱ-starshaped subset $X \subset B$ such that $T(X) \subset X$ and the norm closure $\overline{T(X)}$ is contained in B.*

Proof. If $\operatorname{Fix} T = \{x \in B\colon x = Tx\} \neq \emptyset$ and $y \in \operatorname{Fix} T$, then it is sufficient to take $\{y\}$ in place of X. On the other hand, let X be such a ϱ-starshaped set. Without loss of generality we may assume that $0 \in X$. Then for every $k \in N$ a sequence

$$\{[(1-1/k)T]^n(0)\}$$

is convergent to a point $x_k \in B$ for which we have

$$(1-1/k)Tx_k = x_k$$

(see [1]). It is evident that x_k lies in the norm closure of $T(X)$. Let us take a subsequence $\{x_{k_i}\}$ which converges weakly and let x be its limit. Then x must be less than 1, $\|x_{k_i}\| \leqslant \|Tx_{k_i}\|$ and

$$1 \leqslant \limsup \frac{\sigma(Tx_{k_i}, Tx)}{\sigma(x_{k_i}, x)} \leqslant \sigma(x, Tx)$$

(see Corollary 2). Thus $x = Tx$.

Remark. In fact, the above sequence is strong convergent to the fixed point of T with the smallest norm (see Theorem 13 in [2]).

COROLLARY 4. *Suppose T: $B \to B$ is holomorphic and $\overline{T(B)} \subset B$ (the norm closure is used here). Then T has a fixed point in B and it is unique.*

Proof. Since the set of fixed points of T is affine [2], it must contain only one point by using the fact that $\overline{T(B)} \subset B$.

THEOREM 4. *Suppose T: $B \to B$ is holomorphic with* $\operatorname{Fix} T \neq \emptyset$. *Let X be non-empty, closed in (B, ϱ) and convex in H. If $T(X) \subset X$, then* $\operatorname{Fix} T \cap X \neq \emptyset$.

Proof 1. If $x \in X$, then the sequence $\{T^n x\}$ is ϱ-bounded and its $A(X, \{T^n x\})$ is a fixed point.

Proof 2. If $x \in \operatorname{Fix} T$, then its ϱ-projection on X is a fixed point.

Before proving the next theorem we must show the following lemma:

LEMMA 3. *If $\{\{x_i^m\}: m = 1, 2, \ldots\}$ is a family of ϱ-bounded sequences in which every $\{x_i^m\}$ tends weakly to the same y and*

$$r_m = r_B(\{x_i^m\}) \geqslant r(z, \{x_i^{m+1}\}) \geqslant r_B(\{x_i^{m+1}\}) = r_{m+1}$$

for $m = 1, 2, \ldots$ and a certain $z \in B$, then $z = y$.

Proof. This result follows from the fact that

$$z \in K\left(y, \tanh^{-1}\left(\frac{\tanh^2(r_m) - \tanh^2(r_{m+1})}{1-\tanh^2(r_{m+1})}\right)^{1/2}\right)$$

for every m (see Corollary 3).

THEOREM 5. *Let T: $B \to B$ be a holomorphic mapping and let $\{T^n x\}$ be an iterative sequence. Then $\{T^n x\}$ is weakly convergent to a fixed point of T iff*

$$\sup \|T^n x\| < 1 \quad \text{and} \quad T^{n+1}x - T^n x \rightharpoonup 0.$$

Proof. Let a subsequence $\{T^{n_i}x\}$ be weakly convergent to y. Then, for every m, $\{T^{n_i+m}x\}$ is also weakly convergent to y and

$$r_B(\{T^{n_i+m}x\}) \geqslant r(Ty, \{T^{n_i+m+1}x\}) \geqslant r_B(\{T^{n_i+m+1}x\}).$$

$Ty = y$ is a consequence of Lemma 3. Now it is easy to notice that for every $z = Tz$ we have

$$r(z, \{T^n x\}) = \lim \varrho(T^n x, z),$$

and this completes the proof.

Remark. If in Lemma 3 we have a Banach space with the Opial property [4] and a weakly compact set there, then for sequences in this set Lemma 3 is true. To show this we have to use additionally the theorem of Šmulian (see [5]). Therefore, for a non-expansive mapping which maps this set into itself the analogous theorem to Theorem 5 can be proved.

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