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SMALL SETS IN THE SUPPORT OF A FOURIER-STIELTJES TRANSFORM

BY

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Let $\boldsymbol{T}$ be the circle group and $M(\boldsymbol{T})$ the usual convolution algebra of finite Borel measures on $\boldsymbol{T}$. The *Fourier-Stieltjes transform* of $\mu \in M(\boldsymbol{T})$ is the function $\hat{\mu}$ defined on the integers $\boldsymbol{Z}$ by

$$\hat{\mu}(n) = \int_{\boldsymbol{T}} e^{-in\theta} d\mu(\theta).$$

Write $M_0(\boldsymbol{T}) = \{\mu \in M(\boldsymbol{T}): \hat{\mu} \in C_0(\boldsymbol{Z})\}$.

In [1], Graham has proved the following result:

Suppose that $\mu \in M(\boldsymbol{T})$ and that for each $\varepsilon > 0$ the set

$$S_\varepsilon = \{n \in \boldsymbol{Z}: |\hat{\mu}(n)| \geqslant \varepsilon\}$$

is Sidon. Then $\hat{\mu} \in C_0(\boldsymbol{Z})$.

Actually, Graham proves this result under the milder condition that S_ε satisfies, for some fixed $N \in \boldsymbol{Z}^+$,

$$(\#) \qquad \sup\{\min(|A_1|, |A_2|, \ldots, |A_N|): A_1 + A_2 + \ldots + A_N \subset S_\varepsilon\} < \infty$$

for every ε.

A subset $E \subset \boldsymbol{Z}$ is said to satisfy *lacunarity condition* ($\mathscr{L}$) if for every infinite sequence $\langle m_j \rangle_1^\infty \subset \boldsymbol{Z}$

$$\varliminf_j \{E - m_j\}$$

is finite.

Any set E satisfying

$$\sup\{\min(|A|, |B|): A + B \subset E\} < \infty$$

evidently satisfies lacunarity condition ($\mathscr{L}$). Thus every Sidon set satisfies lacunarity condition ($\mathscr{L}$) (see López and Ross [2], p. 8). Furthermore, any set $E = \{n_1 < n_2 < \ldots\} \subset \boldsymbol{Z}^+$, satisfying the gap condition

$$\lim_{k \to \infty} (n_{k+p} - n_k) = \infty$$

for some fixed $p \in Z^+$, satisfies lacunarity condition ($\mathscr{L}$). We prove the following generalization of [1] (our method is quite different from that of Graham):

THEOREM. *Let $\mu \in M(T)$ and suppose that for every $\varepsilon > 0$ the set S_ε satisfies lacunarity condition ($\mathscr{L}$). Then $\hat{\mu} \in C_0(Z)$.*

Proof. Suppose that $\hat{\mu} \notin C_0(Z)$. We shall force a contradiction: There exists an $\varepsilon_0 > 0$ such that

$$\left|\{n \in Z: |\hat{\mu}(n)| \geqslant \varepsilon_0\}\right| = \infty.$$

Let $0 < \varepsilon < \varepsilon_0$ be given. Then the set S_ε contains S_{ε_0}. Let $\langle m_j \rangle_1^\infty$ be any infinite sequence contained in S_{ε_0}. Let ν be any weak-* cluster point in $M(T)$ of $\exp[-im_j]d\mu$. Assume without loss of generality that $\exp[-im_j]d\mu$ converges weak-* to ν in $M(T)$. By the Helson Translation Lemma ([4], p. 66),

$$\nu \in M_0^\perp(T), \tag{1}$$

where $M_0^\perp(T) = \{\omega \in M(T): \omega \perp \varrho \text{ for each } \varrho \in M_0(T)\}$. Observe that

$$|\hat{\nu}(0)| \geqslant \varepsilon_0 > 0. \tag{2}$$

Inasmuch as

$$\varliminf_j \{S_\varepsilon - m_j\}$$

is finite, we gather that $|\hat{\nu}(m)| < \varepsilon$ for all $m \in Z$ outside a finite set. Thus

$$\nu \in M_0(T). \tag{3}$$

As a consequence of (1) and (3) we may conclude that $\nu = 0$. However, this contradicts (2). The proof is complete.

Comments. (i) An analogue of our Theorem is valid for any non--discrete compact abelian group.

(ii) A simple Riesz product argument shows the existence of a $\mu \in M(T)$ such that each S_ε is a $\Lambda(1)$-set and $\hat{\mu} \notin C_0(Z)$.

(iii) The author has recently obtained a quantitative generalization of the main result of this paper:

Let E satisfy lacunarity condition ($\mathscr{L}$). Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|\mu\| < 1$ and

$$\limsup_{n \notin E} |\hat{\mu}(n)| < \delta,$$

then

$$\limsup_{n \in E} |\hat{\mu}(n)| \leqslant \varepsilon.$$

The relation of δ to ε is as follows: Given $\varepsilon > 0$, choose $r \in Z^+$ $(r \geqslant 31)$ such that $r^{1/2}/4 \geqslant 1/\varepsilon$. Then $\delta = \varepsilon e^{-r}/2$. Notice that δ does not depend on the Sidon constant of E if E is a Sidon set. For a proof of this result the reader is referred to [3].

(iv) We say that $E \subset Z$ satisfies *lacunarity condition* $(\mathscr{L}_1)$ if it satisfies lacunarity condition $(\mathscr{L})$. Having defined lacunarity conditions $(\mathscr{L}_i)$, $i = 1, 2, \ldots, N$, we say that E satisfies *lacunarity condition* $(\mathscr{L}_{N+1})$ if for every infinite sequence $\langle m_j \rangle_1^\infty \subset Z$ the limit

$$\underline{\lim}_{j} (E - m_j)$$

satisfies lacunarity condition $(\mathscr{L}_N)$. Simple modifications in the proof of our Theorem yield the following result:

Suppose $\mu \in M(T)$ and $N \in Z^+$. If for every $\varepsilon > 0$ the set

$$S_\varepsilon = \{m \in Z: |\hat{\mu}(m)| \geqslant \varepsilon\}$$

satisfies lacunarity condition $(\mathscr{L}_N)$, then $\hat{\mu}$ vanishes at infinity.

Notice that any set S_ε satisfying (#) satisfies lacunarity condition $(\mathscr{L}_N)$.

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