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A FUNCTION-THEORETIC PROOF OF A THEOREM OF STAMPFLI

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A complete description of subnormal weighted shifts was first obtained by Stampfli [6]. Then came an "elegant and easy to state" moment sequence description by Berger (see [3]) and Gellar and Wallen [2]. Precisely, this description reads: a weighted shift, with norm one, is subnormal if and only if the squares of the partial products of its weights constitute the moment sequence of a probability measure in the unit interval, with 1 in the support of the measure.

Based on this characterization, Gellar and Wallen [2] provided new proofs of several theorems of Stampfli, including Theorem S below, and more recently Cowen and Long [1] constructed a subnormal Toeplitz operator that is not analytic. The purpose of this brief note is to call attention to a nice analytic function that "interpolates" the moment sequence in the characterization. An exploitation of this function reveals a new (more geometric) proof of a theorem of Stampfli and weakens a necessary and sufficient condition for the unitary equivalence of two subnormal shifts. Further investigation of the connection between this function and subnormal weighted shifts should yield more information about subnormal shifts.

First we describe the function. Let

$$f(z) = \int_0^1 r^z \, d\mu,$$

where μ is a probability measure. Then f is a bounded analytic function on the (open) right half-plane (see, e.g., [5], p. 304) and obviously "interpolates" the moment sequence $\{\int_0^1 r^n \, d\mu\}$. This function is nice because $g(x) = \log f(x)$ is convex; in fact, g is convex on $[0, \infty)$, with $g(0) = 0$ (see [4], p. 156).

We now prove the following theorem of Stampfli:

THEOREM S. *Let S be a weighted shift; that is, $Se_n = w_n e_{n+1}$ for some orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Suppose S is subnormal, with norm* 1. *Then $w_j = w_{j+1}$ for some $j \geq 0$ implies $w_j = w_{j+1} = 1$ for all $j \geq 1$.*

Proof. Let $p_0 = 1$ and $p_n = (w_0 w_1 \dots w_{n-1})^2$ for $n \geqslant 1$. According to the characterization, there is a probability measure μ on $[0, 1]$, with $1 \in \operatorname{support}(\mu)$, such that

$$p_n = \int_0^1 r^n \, d\mu \quad \text{for } n \geqslant 0.$$

Define $f(0) = 1$ and

$$f(z) = \int_0^1 r^z \, d\mu$$

on the right half-plane and

$$g(x) = \log f(x) \quad \text{on } [0, \infty).$$

By hypothesis,

$$p_{j+1} = p_j w_j^2 \quad \text{and} \quad p_{j+2} = p_{j+1} w_{j+1}^2 = p_j w_j^4.$$

Thus

$$g(j) = \log p_j, \quad g(j+1) = \log p_j + 2 \log w_j$$

and

$$g(j+2) = \log p_j + 4 \log w_j;$$

it follows that the three points $(j, g(j))$, $(j+1, g(j+1))$ and $(j+2, g(j+2))$ are collinear. By the convexity of g, $g(x) = ax+b$ for $j \leqslant x \leqslant j+2$. This implies that $f(x) = e^{ax+b}$ on $[j, j+2]$. Since f is analytic, $f(z) = e^{az+b}$ on the right half-plane. But μ is unique, so the only way

$$\int_0^1 r^z \, d\mu = e^{az+b}$$

is when the support of μ is contained in $\{0, e^a\}$ with $\mu(\{e^a\}) = e^b$. Since $1 \in \operatorname{support}(\mu)$, $e^a = 1$. Therefore

$$f(z) = \int_0^1 r^z \, d\mu = e^b.$$

In particular, $f(1) = f(2) = \dots = e^b$; it follows that

$$e^b = w_0^2, \quad w_1 = w_2 = \dots = 1.$$

This completes the proof.

Finally, let S and T be two weighted shifts with weight sequences $\{w_j\}$ and $\{v_j\}$, respectively, and let

$$p_n = (w_0 \dots w_{n-1})^2 \quad \text{and} \quad q_n = (v_0 \dots v_{n-1})^2.$$

It is known that S is unitarily equivalent to T if and only if $p_n = q_n$ for $n \geqslant 1$. For subnormal shifts this condition can be weakened as follows:

PROPOSITION. *Let S and T be subnormal weighted shifts. Then S is unitarily equivalent to T if and only if $p_{n_j} = q_{n_j}$, where $\{n_j\}$ is a subsequence satisfying $\sum 1/n_j = \infty$.*

Proof. The necessity is obvious. The sufficiency is due to Müntz–Szász theorem which states that if $\sum 1/n_j = \infty$, then $\{1\} \cup \{r^{n_j}\}_{j=1}^{\infty}$ spans $C[0, 1]$. This means that the probability measures associated with S and T are the same, whence $p_n = q_n$ for $n \geqslant 1$. Therefore S and T are unitarily equivalent, completing the proof.

Remarks. (1) We point out here that it is the proof of Müntz–Szász theorem [5], p. 304 (which is based on the same analytic function), that leads us to the new proof of Stampfli's theorem.

(2) The weakened condition in the Proposition is not sufficient for two hyponormal shifts to be unitarily equivalent.

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