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ON LEJA-GÓRSKI APPROXIMATIONS IN THE SPATIAL DIRICHLET'S PROBLEM

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1. Let R^3 be the cartesian linear space of points $x = [x^1, x^2, x^3]$, x^i being real numbers ($i = 1, 2, 3$). We write $|x| = (x^1x^1 + x^2x^2 + x^3x^3)^{1/2}$ $\varrho(x, A) = \inf\{|x-y|: y \in A \subset R^3\}$. Let F be a smooth surface in R^3 dividing it into two domains, D and D_∞, D bounded. Let $f(x)$ be a real continuous function of $x \in F$ and $u(x, f)$ the solution of the following Dirichlet's problem: find a u continuous for $x \in R^3 \cup \{\infty\}$ and such that $u(x, f) = f(x)$ $(x \in F)$, $u(\infty, f) = 0$ and $\Delta u = 0$ in $D \cup D_\infty$. The extremal points method provides an effective sequence $u_n = U^{-\sigma_n}(x) + b_n \to u(x, f)(x \in D)$. We have proved in [7] that under suitable assumptions $u_n - u = O(n^{-1/6})$ locally uniformly in D. In the present paper a sharper estimate $u_n - u = O(n^{-1/4})$ is obtained under less restrictive assumptions and by a simpler argument. We refer to [1], [2], [6], [7] for more details about the notions used in this note and for some lemmas.

2. Let μ, ν be any measures. We use classical and special potentials and energies, defined as follows:

$$
\begin{aligned}
U^\mu(x) &= \int \frac{1}{|x-y|} d\mu(y), \quad (\mu, \nu) = \int U^\mu d\nu, \quad \|\mu\|^2 = (\mu, \mu), \\
U_0^\mu(x) &= \int \left(\frac{1}{|x-y|}\right)_0 d\mu(y), \quad (\mu, \nu)_0 = \int U_0^\mu d\nu, \quad \|\mu\|_0^2 = (\mu, \mu)_0.
\end{aligned} \tag{1}
$$

The sign "$_0$" denotes that the definition of the integrand is completed by putting it equal to 0 when $x = y$.

(μ, ν) and $(\mu, \nu)_0$ are symmetric bilinear forms. $\|\mu\|^2 > 0$, even for signed measures, with the sole exception for $\mu = 0$ [2]. This implies $|(\mu, \nu)| \leqslant \|\mu\| \cdot \|\nu\|$. Last symbols denote, of course, square roots of energies. In this connection, see [1].

U_0^μ etc. are primarily destined for measures with atoms.

3. For a reason given in [7], and which appears also from the present section 4, we call a continuous function $f(x)$ $(x \in F)$ *solvable* (in the sense of Siciak, [10]), if it can be written in the form $f(x) = U^{-\varphi}(x)+b$ (φ a positive measure on F with $\varphi(F) = 1$, b a constant). Then

$$(2) \qquad u(x,f) = U^{-\varphi}(x)+b \quad (x\in D).$$

If f is any Lipschitz function, i.e. if

$$(3) \qquad |f(x_1)-f(x_2)| \leqslant c|x_1-x_2| \quad (x_1, x_2\in F,\ c \text{ a constant}),$$

then there exists a positive number λ_0 such that for $\lambda\in\langle 0, \lambda_0\rangle$ the function λf is solvable ([5], as interpreted by [7]). The general case of a Lipschitz f is thus reduced to that of f being both Lipschitz and solvable. We proceed under these assumptions. By a theorem given in [3], p. 213, $u(x,f)$ and consequently $U^{-\varphi}$ is then Lipschitz, i.e.

$$(4) \qquad |u(x_1,f)-u(x_2,f)| \leqslant q|x_1-x_2| \quad (x_1, x_2\in R^3,\ q \text{ a constant})$$

provided F is a C^2 surface (or, more generally, a Liapounoff one).

4. Leja's extremal measure σ_n is defined as an atomic measure with a mass $1/n$ on each of n different points $e_i = e_{i,n-1}\in F$ $(i = 0, 1, \ldots, n-1)$, which are chosen so as to give the minimal value to the expression

$$(5) \qquad I_n \overset{\mathrm{df}}{=} \frac{n}{n-1}\|\sigma_n\|_0^2+2\int f d\sigma_n = \frac{n}{n-1}\sum_{i\neq k}\sum \frac{1}{|e_i-e_k|}\cdot\frac{1}{n^2}+2\sum_{i=0}^{n-1} f(e_i)\cdot\frac{1}{n}.$$

We have

$$(6) \qquad U^{-\sigma_n}(x) = -\sum_{i=0}^{n-1}\frac{1}{|x-e_i|}\cdot\frac{1}{n}\to U^{-\varphi}(x) \quad (x\notin F).$$

The formula for b in (2) may be

$$(7) \qquad b_n\to b, \quad b_n \overset{\mathrm{df}}{=} \frac{n}{n-1}\|\sigma_n\|_0^2+\int f d\sigma_n$$

(for explanation, compare with (5)). These results are essentially due to Górski [4, 5]; see also [7]).

5. We will prove the following theorem:

THEOREM. *Let f be a solvable Lipschitzian function on $F\in C^2$. Then*

$$(8) \qquad |U^{-\delta_n}(x)-U^{-\varphi}(x)| \leqslant \frac{\sqrt{2(1+q)}}{\sqrt{r}}n^{-1/4} \quad \left(x\in D\cup D_\infty,\ r = \varrho(x, F),\ n > (2/r)^2\right),$$

$$(9) \qquad |b_n-b| \leqslant (1+2q)n^{-1/2}+\|\varphi\|\sqrt{1+q}\cdot n^{-1/4} \quad (\textit{for any } n),$$

where q is the constant from (4).

6. Proof. Let ψ_{in} be the measure of mass $1/n$ spread uniformly on the sphere ∂B_i, $B_i = \{x\colon |x-e_i| < n^{-1/2}\}$ $(i = 0, \ldots, n-1)$ and $\psi_n = \psi_{0n}+\psi_{1n}+\ldots+\psi_{n-1,n}$. It was shown in [7], section 8 ((8.4) combined with (6.4)), that

$$\|\psi_n-\varphi\|^2 \leqslant (1+q)n^{-1/2} \qquad (n = 2, 3, \ldots). \tag{10}$$

Let $x_0 \in D \cup D_\infty$ and let γ be the measure of mass 1, spread uniformly on the sphere ∂B, $B = \{x\colon |x-x_0| < \frac{1}{2}r\}$, r from (8). Let $n > (2/r)^2$. Then the potentials below are harmonic in $B \cup \partial B$, and so their values at x_0 are equal to their spherical means on ∂B, i. e. to their integrals with respect to γ:

$$|U^{-\psi_n}(x_0)-U^{-\varphi}(x_0)| = |\int U^{\varphi-\psi_n}d\gamma| = |(\varphi-\psi_n, \gamma)|$$
$$\leqslant \|\varphi-\psi_n\|\cdot\|\gamma\| = \|\varphi-\psi_n\|(r/2)^{-1/2} \leqslant (1+q)^{1/2}n^{-1/4}(r/2)^{-1/2},$$

where the last inequality holds by (10); so (8) is proved.

To obtain (9), integrate (2) with respect to φ. This yields $b = I-\int ud\varphi$, where $I = \|\varphi\|^2+2\int ud\varphi$. By (7) and (5), $b_n = I_n-\int ud\sigma_n$. Remember that $u = f$ on F. We have proved in [7], section 14, that

$$0 \leqslant I-I_n \leqslant (1+q)n^{-1/2} \qquad \big(n = 2, 3, \ldots;\ q \text{ from (4)}\big). \tag{11}$$

Thus, it remains to estimate $\int ud\varphi-\int ud\sigma_n$. Now, by (4),

$$\Big|\int ud\psi_n-\int ud\sigma_n\Big| = \Big|\int u(x,f)d(\psi_n-\sigma_n)\Big| \tag{12}$$
$$= \Big|\sum_{i=0}^{n-1}\int\{u(e_i,f)-u(x,f)\}d\psi_{in}\Big| \leqslant \sum\int q|e_i-x|d\psi_{in} = qn^{-1/2},$$

and by (2) and (10)

$$\Big|\int u(x,f)d\varphi-\int u(x,f)d\psi_n = \Big|\int U^{-\varphi}d(\varphi-\psi_n)\Big| \tag{13}$$
$$= |(-\varphi, \varphi-\psi_n)| \leqslant \|\varphi\|\cdot\|\psi_n-\varphi\| \leqslant \|\varphi\|(1+q)^{1/2}n^{-1/4}.$$

(11), (12) and (13) give us (9), and the proof is completed.

7. To obtain from (8) and (9) numerical bounds, we ought to know q and $\|\varphi\|$. Now, by (2), $\|\varphi\|^2 = \int U^\varphi d\varphi = \int(b-f)d\varphi \leqslant b-\inf f$, so by (9)

$$\|\varphi\|^2 \leqslant b_n-\inf f+(1+2q)n^{-1/2}+(1+q)^{1/2}n^{-1/4}\|\varphi\|. \tag{14}$$

When solving this inequality for $\|\varphi\|$, we obtain an effective bound for it in terms of f and q. So we raise a question, connected also with another one, relative to the density of φ, proposed in [9]:

P 549. Give a bound for q in (4) in terms of F and f only.

8. The derivatives also converge as $n^{-1/4}$. This is easily seen by Poisson's formula. Indeed, let $\varrho(x_0, F) = r > 0$ and $n > (\frac{1}{2}r)^{-2}$ as in (8). Denote by γ the uniform unit measure on $\{y: |y-x_0| = \frac{1}{2}r\}$. Let $U = U^{\sigma_n} - U^{\varphi}$, let D denote the differentiation in any fixed direction and θ the angle of $y-x_0$ with this direction. Then by (8)

$$(15)\qquad |DU(x_0)| = \left|\int U(y)\left[D\frac{|y-x_0|^2-|x-x_0|^2}{|x-y|^3}\right]_{x=x_0} d\gamma(y)\right|$$
$$\leqslant \int |U(y)||-3(\tfrac{1}{2}r)^{-2}\cos\theta|\,d\gamma \leqslant \mathrm{const}\, r^{-3}n^{-1/4}.$$

9. As pointed out in [7], the extremal points method works in the N-dimensional space ($N \geqslant 4$) as well. In (1), $1/|x-y|$ is to be replaced now by $1/|x-y|^{N-2}$, and analoguous changes are to be introduced into (5) and (6). The radii of B_i's (section 6) are now $n^{-1/(N-1)}$, so $\|\psi_{in}\|^2 = n^{-N/(N-1)}$ and we get $\|\psi_n-\varphi\|^2 = (1+q)n^{-1/(N-1)}$ instead of (10). Thus we have

THEOREM. *In the space R^N ($N \geqslant 3$), for $F \in C^2$ and any Lipschitzian solvable f,*

$$(16)\qquad |U^{-\sigma_n}(x) - U^{-\varphi}(x)| \leqslant \frac{\sqrt{2^{N-2}}(1+q)}{\sqrt{r^{N-2}}}\, n^{-1/2(N-1)}$$
$$(x \in D \cup D_\infty,\ r = \varrho(x, F),\ n > (2/r)^{N-1}),$$

$$(17)\qquad |b_n - b| \leqslant (1+2q)n^{-1/(N-1)} + \|\varphi\|\sqrt{1+q}\cdot n^{-1/2(N-1)}$$
$$(n = 2, 3, \ldots)$$

10. Put

$$k = k_n = 1/(\text{average distance between extremal points})$$
$$= n/\Sigma_i \varrho\big(e_i, \{e_q\}_{q\neq i}\big) \qquad (e_q = e_{q,n-1}).$$

The bounds both in (16) and (17) are $O(k^{-1/2})$. In this sense, the degree of convergence of the extremal points method does not depend on the dimension N, with a sligth exception for $N = 2$ — in this case we get $O(k^{-1/2}\log k)$ [8].

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