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REMARKS ON THE PERMEABILITY OF SUBMEASURES ON FINITE ALGEBRAS

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Introduction. Let $\mathscr{A}$ be an algebra of subsets of a set X. A subadditive set function $\varphi\colon \mathscr{A} \to [0, \infty)$ is called a *submeasure* if φ is increasing and $\varphi(\emptyset) = 0$. We follow Bandt [1] and define the *permeability* of a submeasure φ as

$$\alpha(\varphi) = \sup\{\mu(X)\colon \mu \text{ is a finitely additive measure on } \mathscr{A},\ \mu \leqslant \varphi\}.$$

It is useful to express $\alpha(\varphi)$ in terms of multiple coverings of X as follows (see [5]):

$$\alpha(\varphi) = \inf\left\{\frac{1}{k}\sum_{i}^{m} \varphi(A_i)\colon \sum_{i}^{m} 1_{A_i} = k \cdot 1_X\right\}, \tag{1}$$

where $A_i \in \mathscr{A}$ and 1_{A_i} denotes the indicator function of A_i.

In this note we consider the permeability on finite algebras. More precisely, we take into account the numbers α_n defined by

$$\alpha_n = \inf\{\alpha(\varphi)\colon \varphi \text{ is a normalized submeasure (i.e., } \varphi(X_n) = 1) \text{ on } \mathscr{P}(X_n)\},$$

where $\mathscr{P}(X_n)$ is an algebra of all subsets of an n-point set X_n. The interest in these numbers comes from a construction of a pathological submeasure (i.e., a non-trivial submeasure φ with $\alpha(\varphi) = 0$; see [3] and [5]). It is clear that there do not exist pathological submeasures on finite algebras because for any normalized submeasure on $\mathscr{P}(X_n)$ we have $\varphi(\{i\}) \geqslant 1/n$ for some i, so that the measure μ such that $\mu(A) = 1/n$ for $A \supset \{i\}$ and $\mu(A) = 0$ otherwise satisfies $\mu \leqslant \varphi$, $\mu(X_n) = 1/n$. Let us note that $1/n \leqslant \alpha_n$. On the other hand, we take into consideration the normalized submeasure $\varphi(A) = \frac{1}{2}$ for $A \in \mathscr{P}(X_n) \setminus \{\emptyset, X_n\}$. Then if we notice that all $(n-1)$-point sets from $\mathscr{P}(X_n)$ form an $(n-1)$-fold exact covering of X_n, from (1) we obtain

$$\alpha_n \leqslant \alpha(\varphi) \leqslant \frac{n}{2(n-1)}.$$

Topsøe [5] gave the inverse inequality for any normalized symmetric submeasure φ on $\mathscr{P}(X_n)$ (i.e., a submeasure φ for which $\varphi(A)$ depends only on the cardinality of A). However, it can be proved that $\lim \alpha_n = 0$ (see [3] and [5]). The reader can find more information about convergence of α_n's in [1]. In that paper the author followed Vasak and asked:

What is the smallest number q with

$$\alpha_q < \frac{q}{2(q-1)}?$$

Bandt proved $6 \leqslant q \leqslant 11$ and suggested that this number is 11. In the present note we prove $q \leqslant 9$ (see Example 3). It is clear that $\alpha_{n+1} \leqslant \alpha_n$. We suppose that $\alpha_{n+1} < \alpha_n$. If this conjecture is true, then the same example will imply $\alpha_{10} < 10/18$.

An upper bound of q. At the first sight Bandt's example (see [1], Example 2) seems to be incidental. The following method allows us to obtain Bandt's covering as well as a better result. This method is based on some combinatorial idea. To make use of it we have to acquire enough information on the numbers

$$B(k, p, r) = \min\{\operatorname{card}\mathscr{B}\colon \mathscr{B} \subset [B]^p, \forall E \in [B]^r \ \exists A \in \mathscr{B}, A \supset E\},$$

where the natural numbers k, p, r satisfy the inequality $k > p > r$ and B is a k-point set (we define $[B]^m$ as the set of all m-point subsets of B). We are especially interested in $B(k, p, r)$ for small values of k, for instance $k = 10$.

Let $\mathscr{D} \subset [B]^p$ be a family satisfying the above equality. For $\mathscr{D}$ we construct the following $(k \times B(k, p, r))$-matrix:

(1) In k rows of this matrix we write in a sequence all numbers from 1 to $B(k, p, r)$.

(2) We cross out the i-th row number from the j-th column of this matrix if $i \in D_j$, $D_j \in \mathscr{D}$.

The sets given as rows in the resulting matrix form an exact $(k-p)$-fold covering $\mathscr{C}$ of $X_{B(k,p,r)}$. Let us note that any r of them do not cover $X_{B(k,p,r)}$. If some $r+1$ cover $X_{B(k,p,r)}$, then putting the value $1/(r+1)$ on every set from $\mathscr{C}$ we can generate a normalized submeasure φ on $X_{B(k,p,r)}$ as follows:

$$\varphi(A) = \begin{cases} 0 & \text{for } A = \emptyset, \\ \min\{h/(r+1)\colon \bigcup_{i=1}^{h} A_i \supset A, A_i \in \mathscr{C}\} & \text{for } A \neq \emptyset. \end{cases}$$

Using the above-mentioned remarks we obtain from (1) the equality

$$\alpha(\varphi) = \frac{1}{(r+1)} \frac{k}{(k-p)}.$$

EXAMPLE 1. We use the above procedure to the matrix

4	4	3	3	2	2	2	1	1	1	1
7	5	6	5	6	5	3	7	4	3	2
8	6	8	7	7	8	4	8	8	6	5

Taking into consideration the columns of the above matrix as members of $\mathcal{S}$ we obtain Bandt's example (see [1], Example 2).

Some known theorems give fragmentary information about the numbers $B(k, p, r)$. In our consideration we shall use the solution of the following problems:

For what integer n is it possible to form a triple system $S(2, 3, n)$ (quadruple system $S(2, 4, n)$), out of n given elements, in such a way that every pair of elements appears in exactly one triple (quadruple)?

It is clear that a necessary condition for the existence of a system $S(2, 3, n)$ is

$$(n-1) \equiv 0 \pmod 2 \quad \text{and} \quad n(n-1) \equiv 0 \pmod 6.$$

and, respectively, a necessary condition for the existence of a system $S(2, 4, n)$ is

$$(n-1) \equiv 0 \pmod 3 \quad \text{and} \quad n(n-1) \equiv 0 \pmod{12}.$$

Reiss [4] and Hanani [2] proved that these conditions are also sufficient. It is obvious that the system $S(2, m, n)$ consists of $\binom{n}{2}:\binom{m}{2}$ different triples or quadruples, where $m = 3$ or $m = 4$.

EXAMPLE 2. Let us consider the triples of a system $S(2, 3, 9)$ as a family $\mathcal{S}$. Then $B(9, 3, 2) = 12$ and we can construct the (9×12)-matrix as above. The sets given as rows form an exact 6-fold covering $\mathcal{C}$ of X_{12}. It is clear that there are three of them which cover X_{12} and no two do. Putting the value $\frac{1}{3}$ on every set of $\mathcal{C}$ we can generate a normalized submeasure φ on X_{12} with

$$\alpha_{12} \leqslant \alpha(\varphi) \leqslant \frac{1}{3}\cdot\frac{9}{6} = \frac{1}{2} < \frac{n}{2(n-1)} \quad \text{for every } n > 1.$$

Applying repeatedly the above procedure to a system $S(2, 4, 13)$ we obtain a 9-fold covering $\mathcal{C}$ on X_{13} and a normalized submeasure φ with

$$\alpha_{13} \leqslant \alpha(\varphi) \leqslant \frac{1}{3}\cdot\frac{13}{9} = \frac{13}{27} < \frac{1}{2}.$$

Let us note that $n/2(n-1)$ is not a good approximation of α_n for $n \geqslant 12$.

EXAMPLE 3. We use the above method to the matrix

1	6	2	2	6	1	1	1	2
5	3	7	3	7	5	5	5	6
9	8	4	4	8	2	3	4	4
10	10	10	9	9	6	7	8	8

Considering the columns as members of $\mathcal{S}$ we obtain the following 6-fold exact covering $\mathcal{C}$ of X_9:

	2	3	4	5				9
1	2			5		7	8	
1		3		5	6		8	9
1	2			5	6	7		
	2	3	4	5				9
1		3	4			7	8	
1	2		4		6		8	9
1		3	4		6	7		
	2	3			6	7	8	9
			4	5	6	7	8	9

No two rows cover X_9. Assigning the value $\frac{1}{3}$ to every set we can generate a normalized submeasure φ on X_9 with

$$\alpha_9 \leqslant \alpha(\varphi) = \frac{1}{3}\cdot\frac{10}{6} = \frac{10}{18} < \frac{9}{16}.$$

Then Example 3 implies that $q \leqslant 9$.

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