COLLOQUIUM MATHEMATICUM

VOL. XLIX 1984 FASC. 1

UNITARY MULTIPLIERS ON $L^2(G)$

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Let G be a locally compact group with left invariant Haar measure. A bounded operator on $L^p(G)$, $1 \leqslant p < \infty$, is called a *multiplier* if it commutes with all left translations. The algebra of all multipliers on $L^p(G)$ is denoted by $CV^p(G)$. In general, the characterization of $CV^p(G)$ seems to be a very difficult problem (except for the case $p = 1$ in which $CV^p(G) = M(G)$) and is far from being solved, although there are many partial results. Let us recall some of them: Herz [4], Theorem C, shows that if G is an amenable group and $p \leqslant q \leqslant 2$ or $p \geqslant q \geqslant 2$, then $CV^p(G) \subset CV^q(G)$ with contraction of the norm. It follows that the algebra $CV^2(G)$ is the largest one. In [5] Kunze and Stein show that for the group $G = SL(2, \boldsymbol{R})$ any function in $L^p(G)$, $1 \leqslant p < 2$, defines a multiplier on $L^2(G)$ by convolution on the right. A functional characterization of $CV^p(G)$ for amenable groups may be found in [3], [8], and [2].

Using Lamperti's characterization of isometries on L^p-spaces ($p \neq 2$), Parrott [7] and Strichartz [9] proved independently in 1968 that the only isometric multipliers on $L^p(G)$, $p \neq 2$, are scalar multiples of right translations. Of course, this is not the case for $p = 2$.

The aim of this paper is to give a method of decomposition of any unitary multiplier on $L^2(G)$ into the product of a right translation operator and a unitary multiplier of a simple form. To do this we introduce the notion of induced multipliers and give a characterization of them.

Let G be a locally compact group which is countable at infinity and let H be a closed subgroup of G. Fix a rho-function ϱ on G and denote by μ the quasi-invariant measure on G/H associated with ϱ (cf. [1], VII, § 2, or [10], Appendix 1). For a multiplier $T \in CV^p(H)$ let us define an operator $\tilde{T}$ on $L^p(G)$ by

$$\langle \tilde{T}f, g\rangle = \int_{G/H} \Big(\int_{H} T({}_xf\cdot {}_x\varrho^{-1/p})(h)\overline{{}_xg(h)}\cdot {}_x\varrho^{-1/q}(h)\,dh\Big)\,d\mu(\dot{x}),$$

where $f \in L^p(G)$, $g \in L^q(G)$, $1/p + 1/q = 1$, and ${}_xf(y) = f(xy)$. Then $\tilde{T}$ is a multiplier on $L^p(G)$ and the map $CV^p(H) \ni T \to \tilde{T} \in CV^p(G)$ is an isometric and isomorphic embedding of $CV^p(H)$ into $CV^p(G)$. The multiplier $\tilde{T}$ will be called the *induced multiplier* of T on $L^p(G)$ and denoted by $\mathrm{Ind}_H^G T$. One can easily show that if $T \in CV^p(G)$ and T^* denotes its conjugate operator acting on $L^q(G)$, where $1/p+1/q = 1$, then $T^* \in CV^q(G)$ and $\mathrm{Ind}_H^G T^* = (\mathrm{Ind}_H^G T)^*$. In particular, if $T \in CV^2(H)$ and T is hermitian (resp. unitary), then $\mathrm{Ind}_H^G T$ is also hermitian (resp. unitary).

For a function φ in $L^\infty(G)$ let M_φ denote the operator on $L^p(G)$ defined by $M_\varphi f = \varphi f$, $f \in L^p(G)$. Moreover, let $L^\infty(G/H)$ denote the set of all $L^\infty(G)$-functions which are constant on right cosets modulo H.

PROPOSITION. *Let G be a locally compact group, countable at infinity, and let H be a closed subgroup of G. A multiplier $S \in CV^p(G)$ is of the form* $\mathrm{Ind}_H^G T$ *for a $T \in CV^p(H)$ if and only if S commutes with all operators M_φ, $\varphi \in L^\infty(G/H)$.*

Proof. Since every multiplier $\mathrm{Ind}_H^G T$, $T \in CV^p(G)$, commutes (by the definition) with all operators M_φ, $\varphi \in L^\infty(G/H)$, only the converse implication must be shown. Assume S is in $CV^p(G)$ and commutes with all M_φ, $\varphi \in L^\infty(G/H)$. Let s be a Borel section of G fibered by H (cf. [10], p. 374-375) and normalized so that $s(\dot{1}) = 1$. Let t be a function from G to H, defined by $t(x) = s(\dot{x})^{-1}x$. For any continuous function f on H with compact support, any Borel subset E in G/H such that $\mu(E) < \infty$, and for $x \in G$ define functions $\Phi_{f,E}$ and $\Psi_{f,E}$ on G by

$$\Phi_{f,E}(x) = \varrho^{-1/p}(x) f\big(t(x)\big) 1_E(x),$$

$$\Psi_{f,E}(x) = \varrho^{-1/q}(x) f\big(t(x)\big) 1_E(x),$$

where 1_E is the characteristic function of E. We have

$$\Phi_{f,E} \in L^p(G) \quad \text{and} \quad \|\Phi_{f,E}\|_{L^p(G)} = \|f\|_{L^p(H)} \mu(E)^{1/p}$$

and also

$$\Psi_{f,E} \in L^q(G) \quad \text{and} \quad \|\Psi_{f,E}\|_{L^q(G)} = \|f\|_{L^q(H)} \mu(E)^{1/q}.$$

If T is a multiplier on $L^p(H)$ and $\tilde{T}$ denotes the corresponding induced multiplier on $L^p(G)$, then

$$\langle \tilde{T}\Phi_{f,E_1}, \Psi_{g,E_2} \rangle = \mu(E_1 \cap E_2) \langle Tf, g \rangle. \tag{1}$$

Fix a Borel set E_0 in G/H such that $0 < \mu(E_0) < \infty$. For f in $L^p(H)$ and g in $L^q(H)$ the correspondence

$$(f, g) \to \langle S\Phi_{f,E_0}, \Psi_{g,E_0} \rangle$$

is a continuous bilinear form. Thus there exists a bounded operator T on $L^p(H)$ such that

$$\langle Tf, g \rangle = \mu(E_0)^{-1} \langle S\Phi_{f,E_0}, \Psi_{g,E_0} \rangle, \quad f \in L^p(H),\ g \in L^q(H).$$

In fact, T is a multiplier on $L^p(H)$ and we will show that $S = \operatorname{Ind}_H^G T$. The main point of the proof is to show that the definition of the operator T does not depend on the set E_0. To see this, fix an f in $L^p(H)$ and g in $L^q(H)$ and define a set function ν on Borel subsets (of finite measure) in G/H by

$$\nu(E) = \langle S\Phi_{f,E}, \Psi_{g,E}\rangle.$$

For two such sets E_1, E_2 the functions φ_1, φ_2 defined on G by $\varphi_i(x) = 1$ if $\dot{x} \in E_i$ and $\varphi_i(x) = 0$ otherwise, $i = 1, 2$, are constant on right cosets modulo H. Thus

$$(2)\qquad \langle S\Phi_{f,E_1}, \Psi_{g,E_2}\rangle = \langle S(\varphi_1 \cdot \Phi_{f,E_1}), \varphi_2 \cdot \Psi_{g,E_2}\rangle$$
$$= \langle M_{\varphi_1\varphi_2} S\Phi_{f,E_1}, \Psi_{g,E_2}\rangle = \langle S\Phi_{f,E_1\cap E_2}, \Psi_{g,E_1\cap E_2}\rangle = \nu(E_1 \cap E_2).$$

In particular, if $E_1 \cap E_2 = \emptyset$, then $\nu(E_1 \cup E_2) = \nu(E_1)+\nu(E_2)$, and so ν is an additive set function. A routine computation shows that ν is then countably additive. Moreover, since the operator S commutes with left translations and the measure μ is quasi-invariant, ν is also quasi-invariant. Therefore, ν is a multiple of μ, and so

$$\nu(E) = \frac{\mu(E)}{\mu(E_0)} \langle Tf, g\rangle.$$

Now, for any E_1, E_2 by (1) and (2) we get

$$\langle S\Phi_{f,E_1}, \Psi_{g,E_2}\rangle = \frac{\mu(E_1 \cap E_2)}{\mu(E_0)} \langle Tf, g\rangle = \langle \tilde{T}\Phi_{f,E_1}, \Psi_{g,E_2}\rangle,$$

and since the functions of the form $\Phi_{f,E}$ and $\Psi_{g,E}$ ($f \in L^p(H)$, $g \in L^q(H)$, and E is a Borel subset in G/H of finite measure) constitute a linearly dense subset in $L^p(G)$ and $L^q(G)$, respectively, we have $S = \tilde{T}$.

Now we restrict our attention to unitary multipliers on $L^2(G)$. The simplest examples of such operators are right translations or, more precisely, operators R_t, $t \in G$, on $L^2(G)$ of the form

$$(R_t\varphi)(s) = \Delta^{-1/2}(t)\varphi(st), \qquad \varphi \in L^2(G).$$

Another class is formed by multipliers which are induced from unitary multipliers on closed subgroups. We may also compose operators of these two kinds.

THEOREM. *Let G be a locally compact group which is countable at infinity and let U be a unitary multiplier in $CV^2(G)$. Let $\mathscr{A}$ be the set of all functions f in $L^\infty(G)$ such that there exists an $L(f)$ in $L^\infty(G)$ satisfying the equality $U(f\varphi) = L(f)U\varphi$ for all φ in $L^2(G)$. Then*

(a) *there is a uniquely determined closed subgroup H in G such that $\mathscr{A} = L^\infty(G/H)$;*

(b) *there is a* $t \in G$ *such that* $L(f)(s) = f(st)$ *for all* $f \in \mathscr{A}$ *(so that* L *is a right translation)*;

(c) *there is a unitary multiplier* $V \in CV^2(H)$ *such that the multiplier* U *has a decomposition*

$$U = R_t \operatorname{Ind}_H^G V. \tag{3}$$

Proof. By assumption, U is a unitary operator on $L^2(G)$. It follows that if an $L(f)$ exists, then it is unique and the map $f \to L(f)$ is an isometry. One can easily show that $\mathscr{A}$ is then a *-subalgebra in $L^\infty(G)$ and L is a *-homomorphism of $\mathscr{A}$ into $L^\infty(G)$.

Since the *-weak topology for functions f in $L^\infty(G)$ and the weak operator topology for the corresponding operators M_f on $L^2(G)$ coincide, the subalgebra $\mathscr{A}$ is *-weakly closed in $L^\infty(G)$ and L is *-weakly continuous.

Now, since U is a multiplier, L commutes with left translations, and so $\mathscr{A}$ is left translation invariant.

The specified properties of $\mathscr{A}$ together with a result of [6] imply that $\mathscr{A} = L^\infty(G/H)$ for a closed subgroup H, and thus (a) is proved.

To prove (b) observe that

$$L^1(G) * C_0(G/H) = C_0(G/H) \quad \text{and} \quad L^1(G) * L^\infty(G) \subset C(G).$$

Since L is *-weakly continuous and commutes with left translations, it commutes with left convolutions by $L^1(G)$-functions. Thus $f \in C_0(G/H)$ implies $L(f) \in C(G)$. The map

$$C_0(G/H) \ni f \to L(f)(e) \in C$$

is then well defined and determines a non-trivial, continuous, linear and multiplicative functional on $C_0(G/H)$. But then there exists a $t \in G$ such that $L(f)(e) = f(t)$, and since L commutes with left translations, we have also

$$L(f)(s) = f(st), \quad f \in C_0(G/H),\ s \in G. \tag{4}$$

Observe that the right translation by t and L are both *-weakly continuous operators on $L^\infty(G/H)$ and by (4) they coincide on $C_0(G/H)$. But $C_0(G/H)$ is *-weakly dense in $L^\infty(G/H)$, so (4) holds for all f in $L^\infty(G/H)$ and (b) is proved.

Using (a) and (b) we see easily that the unitary multiplier $R_{t^{-1}} U$ commutes with all multiplications by $L^\infty(G/H)$-functions. Thus, by the Proposition, $R_{t^{-1}} U$ is of the form $\operatorname{Ind}_H^G V$ for a unitary multiplier V in $CV^2(H)$, which gives (c).

COROLLARY. *Let* U *be a unitary multiplier on* $L^2(\boldsymbol{R})$, $\boldsymbol{R}$ *being the additive group of the real line. If there exist non-zero functions* f_1, f_2 *in* $L^\infty(\boldsymbol{R})$ *such that* $U(f_1 \cdot \varphi) = f_2 \cdot U\varphi$ *for all* φ *in* $L^2(\boldsymbol{R})$, *then* U *is a convolution by a discrete,* *locally bounded measure with support contained in an arithmetic progression in* $\boldsymbol{R}$.

Proof. Since f_1 is non-constant, the subgroup H in (a) is proper, and so $H = \alpha Z$ for an $\alpha \in \boldsymbol{R}$. By (c) the multiplier U is of the form $R_t \operatorname{Ind}_{\alpha Z}^{\boldsymbol{R}} V$, where V is a multiplier on the discrete group αZ. Thus V is a convolution by a locally bounded measure μ (say) on αZ, and so U is a convolution by its translation $R_t \mu$.

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Reçu par la Rédaction le 2. 10. 1980