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CURVATURE OF LIFT SPACES

BY

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1. Introduction. A triple (M, G, Q) is called a *lift space* of class C^k, $k \geqslant 0$ (see [2]) if M is a connected and paracompact C^∞-manifold, G is a Lie group and Q is a C^k-map from TM into $\boldsymbol{G}$ (the Lie algebra of G). The lift space (M, G, Q) is called *linear* if Q is linear on each fibre of TM.

Suppose we are given a lift space (M, G, Q). For any piecewise C^1-path γ_t, $t \in [a, b]$, in M, the *lift of* γ *with origin* $g \in G$ is the unique path $\bar{\gamma}(g, t)$, $t \in [a, b]$, in G such that

$$\dot{\bar{\gamma}}(g, t) = Q(\dot{\gamma}_t)\bar{\gamma}(g, t), \qquad \bar{\gamma}(g, a) = g \tag{1}$$

(for the existence, see [3], p. 69). The lift space (M, G, Q) is called *flat* (*locally flat*) if the lift with origin e (e is the unit of G) of any closed piecewise C^1-path (of any closed piecewise C^1-path homotopic to zero) is closed. The holonomy group $|K_{Q,p}|$ (the restricted holonomy group $|K^0_{Q,p}|$) at $p \in M$ is the subgroup of G generated (algebraically) by all elements of the form $\bar{\gamma}(e, b)$, where γ_t, $t \in [a, b]$, are piecewise C^1-loops (piecewise C^1-loops homotopic to zero) at p. For any $p \in M$, the well-defined set

$$Q(M, p) = \dot{\bigcup_{q \in M}} \bar{\gamma}^q(e, 1)|K_{Q,p}| \tag{2}$$

(where γ^q_t, $t \in [0, 1]$, is any piecewise C^1-path joining p to q) is called the *holonomy bundle* at p.

In [2] it is proved that the holonomy group of a lift space (M, G, Q) of class C^k, $k \geqslant 0$, at $p \in M$ is a Lie subgroup (with countably many components) of G and the restricted holonomy group at p is its identity component ([2], Proposition 1), while the holonomy bundle at p is a reduced C^{k+1} principal fibre subbundle of $M \times G$ ([2], Corollary 5).

The Lie algebra of $|K_{Q,p}|$ is called the *holonomy algebra* at $p \in M$ and is denoted by $k_{Q,p}$.

In Section 3 of this paper we prove that the holonomy groups and holonomy bundles may be defined by using only piecewise C^∞-paths. This may be viewed as a generalization (in some sense) of a result due to Nomizu and Ozeki [4].

Section 4 contains some infinitesimal factorization property for lift spaces.

By a theorem of Ambrose and Singer [1], the holonomy algebra $k_{Q,p}$ of a *linear* lift space (M, G, Q) of class C^∞ is equal to the subspace of G induced by the curvature form of the corresponding connection $\overline{T}^Q$ (see [2], Remark 4) on $Q(M, p)$. There arises a natural question of the existence of a "curvature" for any lift space of class C^k, $k \geqslant 0$ (not necessarily linear), which measures its deviation from local flatness.

In Section 5 we give an affirmative answer to this question (Proposition 3). Namely, we define a certain curvature quantity for arbitrary lift spaces of class C^k, $k \geqslant 2$, which satisfies a generalization of the Ambrose and Singer result. In particular, its vanishing is equivalent to local flatness.

Throughout this paper, for a left Lie group action $G \times M \to M$ we use the notation

$$Xp = \frac{d}{dt}\bigg|_{t=0} \exp(tX)p, \quad p \in M, X \in G.$$

A similar convention is adopted for right actions. For Lie subgroups $L \subset K$ and $K \subset G$ we shall denote the natural projection $G \to G/K$ (the set of *left* cosets) by φ_K, while the induced map $G/L \to G/K$ will always be denoted by pr. The set of all C^k cross-sections of a C^l vector bundle V over M $(k \leqslant l)$ will be denoted by $\Gamma^k(V)$.

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2. Preliminaries. Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$. In the sequel we shall need the following lemmas:

LEMMA 1 (see [2], Proposition 3 and Corollary 4). *Let $p \in M$. Then:*

(i) *There exists a unique C^{k+1}-map $\Phi_{Q,p}\colon M \to G/|K_{Q,p}|$ such that*

$$(\Phi_{Q,p})_*(v) = Q(v)\Phi_{Q,p}(q) \text{ for } q \in M,\ v \in T_qM, \quad \Phi_{Q,p}(p) = \varphi_{|K_{Q,p}|}(e). \tag{3}$$

(ii) *Let K be a Lie subgroup of G. If there exists a differentiable map $\Phi\colon M \to G/K$ which satisfies* (3), *then $|K_{Q,p}| \subset K$, $\Phi = \mathrm{pr}\,\Phi_{Q,p}$ and*

$$\Phi(\gamma_t) = \varphi_K \bar{\gamma}(e, t), \quad t \in [a, b], \tag{4}$$

for any piecewise C^1-path γ_t, $t \in [a, b]$, in M started from p.

LEMMA 2 (see [2], (ii) of Proposition 4). *The holonomy bundle $Q(M, p)$ at $p \in M$ is equal to the restriction of the principal fibre bundle*

$$\mathrm{id}_M \times \varphi_{|K_{Q,p}|}\colon M \times G \to M \times (G/|K_{Q,p}|)$$

to the graph of the map $\Phi_{Q,p}$.

LEMMA 3 (see [2], Lemmas 6 and 7). *Let $p \in M$ and $(q, g) \in Q(M, p)$. Then*

(i) $\operatorname{ad}(g)|K_{Q,p}| = |K_{Q,q}|$,

(ii) $Q(M,p)g^{-1} = Q(M,q)$.

By the *holonomy algebra bundle* [2] we mean the set

$$k_Q(M) = \bigcup_{q \in M} \{q\} \times k_{Q,q}.$$

LEMMA 4 (see [2], Corollary 6). *The holonomy algebra bundle $k_Q(M)$ is a C^{k+1} vector subbundle of $M \times G$.*

A piecewise C^1-path $(\gamma_t, \bar{\gamma}_t)$, $t \in [a,b]$, in $M \times G$ is called *horizontal* [2] if $\bar{\gamma}_t = \bar{\gamma}(g,t)$ for some $g \in G$. A vector tangent to $M \times G$ is called *horizontal* if it is tangent to a horizontal path. The set of all horizontal vectors will be denoted by $\bar{T}^Q$. In view of (1), the set of all horizontal vectors at $(q,g) \in M \times G$ is given by

$$\bar{T}^Q_{(q,g)} = \{(v, Q(v)g) : v \in T_q M\}. \tag{5}$$

Let $p \in M$. By (2), $M \times G$ is a disjoint union of right cosets $Q(M,p)g$, $g \in G$. Define a distribution D^Q on $M \times G$ by setting

$$D^Q_{(q,g)} = T_{(q,g)}(Q(M,p)g_1), \tag{6}$$

where $(q,g) \in Q(M,p)g_1$ for $g_1 \in G$. By (ii) of Lemma 3, D^Q does not depend on the choice of $p \in M$ and $D^Q_{(q,g)} = D^Q_{(q,e)}g$, $(q,g) \in M \times G$.

LEMMA 5. *The distribution D^Q is of class C^k and*

$$D^Q_{(q,g)} = \bar{T}^Q_{(q,g)} + k_{Q,q}g, \quad (q,g) \in M \times G. \tag{7}$$

Proof. Fix a point $p \in M$. Using local C^{k+1} cross-sections

$$\Pi_l : U_l \ni q \mapsto (q, \pi_l(q)) \in Q(M,p)$$

of the bundle $Q(M,p)$, we have

$$D^Q_{(q,g)} = \big(T_{\Pi_l(q)}(Q(M,p))\big)\pi_l(q)^{-1}g \quad \text{for } (q,g) \in U_l \times G.$$

Since $Q(M,p)$ is a C^{k+1}-submanifold of $M \times G$, D^Q is of class C^k.

Now, let $\xi_t = (\gamma_t, \beta_t)$, $t \in [-1,1]$, be a C^1-path in $Q(M,p)$ and let $\xi_0 = (p,e)$. Then there exists a C^1-path s_t, $t \in [-1,1]$, in $|K_{Q,p}|$ such that $\beta_t = \bar{\gamma}(e,t)s_t$ for each $t \in [-1,1]$. Hence, by (1) and the Leibniz formula, $\dot{\xi}_0 = (\dot{\gamma}_0, Q(\dot{\gamma}_0) + \dot{s}_0)$, so that $D^Q_{(p,e)} = \bar{T}^Q_{(p,e)} + k_{Q,p}$, which implies (7). This completes the proof.

LEMMA 6. *Let ξ_t, $t \in [a,b]$, be a piecewise C^1-path in $M \times G$ and let $\xi_a = (q,g)$. If $\dot{\xi}_t \in D^Q$ for each $t \in [a,b]$, then the entire path ξ_t lies in $Q(M,q)g$.*

Proof. Let $\xi_t = (\gamma_t, \beta_t g)$, $t \in [a,b]$. By (2), we have only to show that $s_t = \bar{\gamma}(e,t)^{-1}\beta_t \in |K_{Q,q}|$ for $t \in [a,b]$. In fact, applying the Leibniz formula to the path β_t, we have

$$\dot{\beta}_t = \dot{\bar{\gamma}}(e,t)s_t + \bar{\gamma}(e,t)\dot{s}_t = Q(\dot{\gamma}_t)\beta_t + \bar{\gamma}(e,t)\dot{s}_t.$$

By (7), we obtain $\bar{\gamma}(e,t)\dot{s}_t \in k_{Q,\gamma_t}\beta_t$. In view of (i) of Lemma 3 we get $k_{Q,\gamma_t}\beta_t = \bar{\gamma}(e,t)k_{Q,q}s_t$. Thus $\dot{s}_t \in k_{Q,q}s_t$ and we obtain $s_t \in |K_{Q,q}|$, $t \in [a,b]$, since $s_a = e$. This completes the proof.

LEMMA 7. *Let $p \in M$, $v_1, v_2, v \in T_pM$ and $a \in R$. Then the vectors $Q(v_1+v_2)-Q(v_1)-Q(v_2)$ and $aQ(v)-Q(av)$ belong to $k_{Q,p}$.*

Proof. By (5) and Lemma 5, $D^Q_{(p,e)} = \{(v, Q(v)+X): v \in T_pM, X \in k_{Q,p}\}$ is a vector subspace of $T_pM \times G$, which immediately implies our assertion.

3. Paths used in the definition of holonomy groups. Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$, and let $p \in M$. In the definition of holonomy groups (restricted holonomy groups) we used piecewise C^1-paths. If we denote by $|K_{Q,p}|_s$ ($|K^0_{Q,p}|_s$) the groups obtained in this way from piecewise C^k-paths, $1 \leqslant s \leqslant \infty$, then we get the following sequence of Lie subgroups (cf. [1], proof of Proposition 1):

$$(8) \qquad |K_{Q,p}|_\infty \subset |K_{Q,p}|_{s+1} \subset |K_{Q,p}|_s, \quad |K^0_{Q,p}|_\infty \subset |K^0_{Q,p}|_{s+1} \subset |K^0_{Q,p}|_s, \quad s \geqslant 1.$$

Similarly, for holonomy bundles associated with these groups we have the principal bundle inclusions

$$(9) \qquad Q(M,p)_\infty \subset Q(M,p)_{s+1} \subset Q(M,p)_s, \quad s \geqslant 1.$$

PROPOSITION 1. *Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$. Then, for every $p \in M$,*

(i) $|K_{Q,p}|_s = |K_{Q,p}|$,

(ii) $|K^0_{Q,p}|_s = |K^0_{Q,p}|$,

(iii) $Q(M,p)_s = Q(M,p)$, $1 \leqslant s \leqslant \infty$.

Proof. By (8) and (9), we have only to show that the proposition is true for $s = \infty$. Put $|K_{Q,p}|_\infty = K$ and define the map $\Phi: M \to G/K$ by $\Phi(q) = \varphi_K(\bar{\gamma}^q(e,1))$, where γ^q_t, $t \in [0,1]$, is a piecewise C^∞-path in M joining p to q in M. It is easy to see that Φ is well defined (cf. [2], proof of (i) of Proposition 3). For any piecewise C^∞-path γ_t in M starting from p we have $\Phi(\gamma_t) = \varphi_K(\bar{\gamma}(e,t))$ and, by (1),

$$\frac{d}{dt}\Phi(\gamma_t) = Q(\dot{\gamma}_t)\Phi(\gamma_t).$$

Thus Φ is differentiable and satisfies condition (3), so that by (ii) of Lemma 1 we obtain $|K_{Q,p}| \subset K$ and, for its identity components, $|K^0_{Q,p}| \subset |K^0_{Q,p}|_\infty$. Therefore, (i) and (ii) hold, which easily implies (iii). This completes the proof.

Remark 1. In the case where M is an analytic manifold, we can still define the groups $|K_{Q,p}|_\omega$, $|K^0_{Q,p}|_\omega$ and the bundle $Q(M,p)_\omega$ by using only piecewise analytic paths in M. The same argument shows that $|K_{Q,p}|_\omega = |K_{Q,p}|$, $|K^0_{Q,p}|_\omega = |K_{Q,p}|$ and $Q(M,p)_\omega = Q(M,p)$.

4. Factorizations of lift spaces. Let (M, G, Q_i), $i = 1, 2$, be lift spaces of class C^k, $k \geqslant 0$. Then (M, G, Q_1) is said to be a *factorization* of (M, G, Q_2) if there exists $q \in M$ such that $|K_{Q_1,q}| \subset |K_{Q_2,q}|$ and the diagram

$$\begin{tikzcd} M \arrow[r, "\Phi_{Q_1,q}"] \arrow[dr, "\Phi_{Q_2,q}"'] & G/|K_{Q_1,q}| \arrow[d, "\mathrm{pr}"] \\ & G/|K_{Q_2,q}| \end{tikzcd}$$

commutes. By (i) of Lemma 1 and (i) of Lemma 3, if there exists such a $q \in M$, then this property holds for every $q \in M$.

Now we give an infinitesimal factorization property.

PROPOSITION 2. *Let (M, G, Q_i), $i = 1, 2$, be lift spaces of class C^k, $k \geqslant 0$. Then the following properties are equivalent:*

(i) *(M, G, Q_1) is a factorization of (M, G, Q_2).*

(ii) *$Q_1(M, p) \subset Q_2(M, p)$ for some point $p \in M$.*

(iii) *$T^{Q_1} \subset D^{Q_1}$.*

(iv) *$Q_1(v) - Q_2(v) \in k_{Q_2,q}$ for each $q \in M$ and $v \in T_qM$.*

Proof. For any piecewise C^1-path γ_t in M, we denote by $\bar{\gamma}^i(g, t)$ the lift of γ with origin $g \in G$ in the lift space (M, G, Q_i), $i = 1, 2$.

(i) $\Rightarrow$ (ii) follows from Lemma 2.

(ii) $\Rightarrow$ (iii). If $Q_1(M, p) \subset Q_2(M, p)$, then by (6) and (7) we have $T^{Q_1} \subset D^{Q_1} \subset D^{Q_2}$.

(iii) $\Rightarrow$ (ii). Let ξ_t, $t \in [a, b]$, be a horizontal path in $M \times G$ and let $\xi_a = (p, e)$. Then $\dot{\xi}_t \in T^{Q_1} \subset D^{Q_2}$, so by Lemma 6 we have $\xi_t \in Q(M, p)$ for $t \in [a, b]$. Thus we obtain $|K_{Q_1,p}| \subset |K_{Q_2,p}|$ and $\xi_t|K_{Q_1,p}| \subset Q_2(M, p)$ for each $t \in [a, b]$, which yields $Q_1(M, p) \subset Q_2(M, p)$.

(ii) $\Rightarrow$ (i). Let $Q_1(M, p) \subset Q_2(M, p)$. Then $|K_{Q_1,p}| \subset |K_{Q_2,p}|$ and, by (2), for any piecewise C^1-path γ_t in M starting from p there exists a piecewise C^1-path s_t in $|K_{Q_2,p}|$ such that $\bar{\gamma}^1(e, t) = \bar{\gamma}^2(e, t)s_t$. Thus, by (4),

$$\mathrm{pr}\,\Phi_{Q_1,p}(\gamma_t) = \mathrm{pr}\big(\bar{\gamma}^1(e, t)\varphi_{|K_{Q_1,p}|}(e)\big) = \bar{\gamma}^2(e, t)\varphi_{|K_{Q_2,p}|}(e) = \Phi_{Q_2,p}(\gamma_t).$$

Hence $\mathrm{pr}\,\Phi_{Q_1,p} = \Phi_{Q_2,p}$.

(iii) $\Leftrightarrow$ (iv) follows from (7). This completes the proof.

Examples. (i) Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$. By Lemma 4 and paracompactness of M, there exists a C^k vector subbundle $B(M) \subset M \times G$ such that $M \times G = B(M) \oplus k_Q(M)$. Let $Q = Q^B + Q^k$ be the corresponding decomposition of Q. Using Lemma 7 we see that, for any $p \in M$, $v_1, v_2, v \in T_pM$ and $a \in R$, the vectors $Q^B(v_1 + v_2) - Q^B(v_1) - Q^B(v_2)$ and $Q^B(av) - aQ^B(v)$ are equal to zero. Thus the lift space (M, G, Q^B)

is linear and of class C^k and, by (iv) of Proposition 2, it is a factorization of (M, G, Q).

(ii) Let $G\times M \to M$ be a left transitive Lie group action and F_M: $M\times G \to TM$ the induced map such that $F_M(q, X) = Xq$ for $q \in M$, $X \in G$. Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$, along F_M [2] (i.e. $F_M(q, Q(v)) = v$, $q \in M$, $v \in T_qM$). We have $k_Q(M) \subset \ker F_M$ (cf. [2], (iv) of Proposition 6). Thus, by (iv) of Proposition 2, every factorization of (M, G, Q) is a lift space along F_M. In particular, given a C^k vector subbundle $B(M) \subset M\times G$ such that $M\times G = B(M)\oplus k_Q(M)$, the induced factorization (M, G, Q^B) of (M, G, Q) (cf. Example (i)) is a linear lift space of class C^k along F_M.

(iii) Let (M, G, Q) be a lift space of class C^k, $k \geqslant 0$, and let $p \in M$. Then every C^k-connection T on the bundle $G \to G/|K_{Q,p}|$ induces a linear factorization of (M, G, Q). In fact, let $Q_1 : T(G/|K_{Q,p}|) \to G$ be the linear map associated with the connection T (cf. [2], Remark 8) and let $Q^T = Q_1\circ(\Phi_{Q,p})_*$. Define a C^k vector subbundle $B(M) \subset M\times G$ by setting

$$B(M)_q = \{Q_1(X\Phi_{Q,p}(q)) : X \in G\}, \quad q \in M.$$

We have $M\times G = B(M)\oplus k_Q(M)$ and, by (3), $Q^T = Q^B$, where $Q = Q^B+Q^k$ is the corresponding decomposition of Q. Thus, by Example (i), the lift space (M, G, Q^T) is a linear factorization of (M, G, Q) of class C^k.

5. Curvature. Let (M, G, Q) be a lift space of class C^k, $k \geqslant 2$. The map

$$K : \Gamma^\infty(TM)\times\Gamma^\infty(TM) \to \Gamma^{k-1}(M\times G)$$

defined by

$$K(X, Y) = [Q(X), Q(Y)]+XQ(Y)-YQ(X)-Q([X, Y]) \tag{10}$$

for $X, Y \in \Gamma^\infty(TM)$ is called the *curvature* of (M, G, Q).

For any map $S : M \to TM\times G$ such that $S(q) \in T_qM\times G$, $q \in M$, we denote by S^* the corresponding (right) G-invariant vector field on $M\times G$.

LEMMA 8. *If $X, Y \in \Gamma^\infty(TM)$, then $K(X, Y) \in \Gamma^{k-1}(k_Q(M))$.*

Proof. Since $K(X, Y) \in \Gamma^{k-1}(M\times G)$ and $k_Q(M) \subset M\times G$ is a C^{k+1} vector subbundle (cf. Lemma 4), we have only to show that $K(X, Y)_q \in k_{Q,q}$ for each $q \in M$. By Lemmas 5 and 6, the distribution D^Q is of class C^k and it is involutive. Thus, for any $q \in M$,

$$\begin{aligned} D^Q_{(q,e)} \ni [X+Q(X)^*, Y+Q(Y)^*]_{(q,e)} \\ = [X, Y]_q+X_qQ(Y)-Y_qQ(X)+[Q(X_q), Q(Y_q)] \\ = [X, Y]_q+Q([X, Y]_q)+K(X, Y)_q, \end{aligned}$$

and from (5) and (7) we obtain $K(X, Y)_q \in k_{Q,q}$, which completes the proof.

Remark 2. If (M, G, Q) is linear, then its curvature is bilinear over the ring of C^∞-functions on M. In this case, given a chart on M with base vectors ∂_i and a basis E_r of G and setting $K(\partial_i, \partial_j) = K^r_{ij}E_r$ and $Q(\partial_i) = Q^r_i E_r$, we have

$$K^r_{ij} = \partial_i Q^r_j - \partial_j Q^r_i + Q^u_i Q^w_j C^r_{uw},$$

where C^r_{uw} are the structure constants of G given by $[E_u, E_w] = C^r_{uw}E_r$.

PROPOSITION 3 (holonomy theorem). *Let (M, G, Q) be a lift space of class C^k, $k \geqslant 2$, and let $p \in M$. Then the holonomy algebra $k_{Q,p}$ is equal to the vector subspace of G spanned by all elements of the form*

$$g^{-1}\big(K(X, Y)_q\big)g, \tag{11}$$

where $X, Y \in \Gamma^\infty(TM)$ and $(q, g) \in Q(M, p)$.

Proof. Let h denote the vector subspace of G spanned by all elements of the form (11). In view of Lemma 8, $K(X, Y)_q \in k_{Q,q}$, so that, by (i) of Lemma 3, we have $g^{-1}\big(K(X, Y)_q\big)g \in k_{Q,p}$ for any $X, Y \in \Gamma^\infty(TM)$ and $(q, g) \in Q(M, p)$. Thus $h \subset k_{Q,p}$. By definition, h is invariant by ad $|K_{Q,p}|$ and, consequently, the connected Lie subgroup H of G with Lie algebra h is a normal subgroup of $|K^0_{Q,p}|$. We must only show that $K = |K^0_{Q,p}|$.

Let $c_M : \bar{M} \to M$ be the universal covering map and let $(\bar{M}, G, \bar{Q})$ be the induced lift space, where $\bar{Q} = Q\circ(c_M)_*$. Let $\bar{p} \in c_M^{-1}(p)$. It is easy to see that $|K_{\bar{Q},\bar{p}}| = |K^0_{Q,p}|$ (cf. [2], Example 6) and the Lie algebra constructed in this way for $(\bar{M}, G, \bar{Q})$ is equal to h, so that we may assume that M is simply connected and that $|K_{Q,p}| = |K^0_{Q,p}|$.

Thus we obtain the principal bundle $\mathrm{pr} : G/H \to G/|K_{Q,p}|$ with structure group $G_1 = |K_{Q,p}|/H$. Let W denote the restriction of the induced principal bundle

$$\mathrm{id}_M \times \mathrm{pr} : M \times (G/H) \to M \times (G/|K_{Q,p}|)$$

to the graph $\mathrm{gr}\,\Phi_{Q,p}$ of the map $\Phi_{Q,p}$. In view of Lemma 2 we have

$$W = \{(q, \varphi_H(g)) : (q, g) \in Q(M, p)\}$$

and, by (i) of Lemma 1, it is a C^{k+1} principal bundle over $\mathrm{gr}\,\Phi_{Q,p}$. Define the subset T of TW by

$$T_{(q,\varphi_H(g))} = \{v + Q(v)\varphi_H(g) : v \in T_qM\} \quad \text{for } (q, g) \in Q(M, p).$$

We assert that T is a flat connection of class C^k on the bundle $W \to \mathrm{gr}\,\Phi_{Q,p}$.

In fact, let us first prove that T is a distribution on W. By (10) we have

$$Q([X, [Y, Z]]) = [Q(X), Q([Y, Z])] + XQ([Y, Z]) - \\ -[Y, Z]Q(X) - K(X, [Y, Z]) \tag{12}$$

for $X, Y, Z \in \Gamma^\infty(TM)$. Summing (12) cyclically in X, Y, Z and using (10) and the Jacobi identity, we obtain

$$(13)\quad \operatorname{cycl} Q\big([X,[Y,Z]]\big) = -\operatorname{cycl}\big(XK(Y,Z)+K(X,[Y,Z])+[Q(X),K(Y,Z)]\big).$$

Define a vector field $V \in \Gamma^k(TW)$ by

$$V_{(q,\varphi_H(g))} = X_q + Q(X_q)\varphi_H(g) \quad \text{for } (q,g) \in Q(M,p).$$

Let Ψ denote the restriction of the map $\mathrm{id}_M \times \varphi_H\colon M \times G \to M \times (G/H)$ to $Q(M,p)$. Clearly, the vector fields V and $(X+Q(X)^*)|_{Q(M,p)}$ are Ψ-related and $K(Y,Z)^*|_{Q(M,p)}$ is Ψ-related with zero, which yields

$$\begin{aligned}(14)\quad & X_qK(Y,Z)\varphi_H(g)+[Q(X_q),K(Y,Z)_q]\varphi_H(g)\\ &= [X+Q(X)^*, K(Y,Z)^*]_{(q,e)}\varphi_H(g) = [X+Q(X)^*, K(Y,Z)^*]_{(q,g)}\varphi_H(e)\\ &= \Psi_{*,(q,g)}\big([X+Q(X)^*, K(Y,Z)^*]_{(q,g)}\big) = [V,0]_{(q,\varphi_H(e))} = 0.\end{aligned}$$

By the definition of h, $K(X,[Y,Z])_q\varphi_H(g) = 0$ for $(q,g) \in Q(M,p)$, which together with (14) applied to (13) gives

$$(15)\qquad \operatorname{cycl} Q\big([X,[Y,Z]]_q\big)\varphi_H(g) = 0, \quad (q,g) \in Q(M,p).$$

Let $(q,g) \in Q(M,p)$ and $v_1, v_2 \in T_qM$. Then there exist X, Y, Z in $\Gamma^\infty(TM)$ such that $v_1 = [X,[Y,Z]]_q$, $v_2 = [Y,[Z,X]]_q$, so by (15) and the Jacobi identity we obtain

$$Q(v_1+v_2)\varphi_H(g) = Q(v_1)\varphi_H(g)+Q(v_2)\varphi_H(g).$$

Thus $T_{(q,\varphi_H(g))}$ is a vector subspace of $T_{(q,\varphi_H(g))}W$ for $(q,g) \in Q(M,p)$, since Q is continuous. Clearly, $\dim T = \dim M$, and using local cross-sections of the bundle $G/H \to G/|K_{Q,p}|$ we see that T is a C^k-distribution on W. Since T is G_1-invariant and $\mathrm{pr}_* T = TM$, it is a C^k-connection on the bundle $W \to \mathrm{gr}\,\Phi_{Q,p}$. By (10) and an easy computation, we verify that the bracket of any two horizontal vector fields on W is horizontal, which shows that T is flat.

By our assumption, M is simply connected and, consequently, T induces a global C^{k+1} cross-section Ξ of the bundle $W \to \mathrm{gr}\,\Phi_{Q,p}$ such that $\Xi(p, \Phi_{Q,p}(p)) = (p, \varphi_H(e))$ and, since $\mathrm{pr}\,\varphi_H(g) = \Phi_{Q,p}(q)$, $(q,g) \in Q(M,p)$, we have

$$(16)\quad \Xi_*\big(v+Q(v)\Phi_{Q,p}(q)\big) = v+Q(v)\Lambda\big(q,\Phi_{Q,p}(q)\big), \qquad q \in M,\ v \in T_qM,$$

where $\Lambda = \mathrm{pr}_W \Xi$ and pr_W is the projection $W \subset M \times (G/H) \to G/H$.

Let $\Phi = \Lambda \circ (\mathrm{id}_M, \Phi_{Q,p})$. From (16) we obtain $\Phi_*(v) = Q(v)\Phi(q)$ for $q \in M$ and $v \in T_qM$. Thus $\Phi\colon M \to G/H$ satisfies condition (3) and, in view of (ii) of Lemma 1, we obtain $|K^0_{Q,p}| \subset H$. This completes the proof.

Using Proposition 1 we easily obtain (cf. [2], (iii) of Lemma 1)

COROLLARY 1. *The holonomy algebra $k_{Q,p}$ at p is equal to the vector subspace of G spanned by all elements of the form*

$$\bar{\gamma}(e, b_\gamma)^{-1}\big(K(X, Y)_q\big)\bar{\gamma}(e, b_\gamma),$$

where $X, Y \in \Gamma^\infty(TM)$, $q \in M$ and γ_t, $t \in [a_\gamma, b_\gamma]$, is a piecewise C^∞-path in M joining p to q.

As an immediate consequence of Proposition 3 we obtain

COROLLARY 2. *A lift space of class C^2 is locally flat if and only if its curvature vanishes.*

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