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UNBOUNDED MULTIPLIERS AND SUMMATION OF SERIES

BY

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Let A be a (sequence to sequence) matrix method of summation. If the sequence of partial sums $\{\sum_1^n u_k\}$ is in the convergence domain of the matrix, then A sums the infinite series $\sum_k u_k$.

We consider the question:

If a regular matrix sums the series $\sum u_k$, does there exist an unbounded sequence $\{\lambda_k\}$ such that A sums $\sum \lambda_k u_k$?

Bryant showed that the answer is yes if A is the $(C, 1)$ matrix ([1], lemma 2.1). More generally, the answer is yes if $\sum u_k$ has bounded partial sums. An example is also given which shows that such a sequence does not necessarily exist if the series does not have bounded partial sums. Finally, an application of theorem 1 is given.

If A is a matrix, c_A is the convergence domain of A. For a sequence x define $\nabla x_k = x_k - x_{k-1}$ (where $x_0 = 0$) and $\nabla x = \{\nabla x_k\}$; then x is the sequence of partial sums of the series whose terms form the sequence ∇x. Since $\sum \nabla x$ is summed by A if and only if $x \in c_A$, we define $\nabla c_A = \{\nabla x : x \in c_A\}$. If x is summed by A to a, this is indicated by $\lim_A x = a$.

The spaces of bounded, convergent and null sequences are denoted by m, c and c_0, respectively, and for $x \in m$,

$$\|x\| = \sup\{|x_n| : n = 1, 2, \ldots\}.$$

For a matrix A, $\|A\| = \sup\{\sum_k |a_{nk}| : n = 1, 2, \ldots\}$.

An index sequence $Q = \{q_i\}$ is a strictly increasing sequence of integers such that $q_0 = 0$. If Q is an index sequence we define

$$Q(j) = \{q_{j-1}+1, \ldots, q_j\} \quad \text{for } j = 1, 2, \ldots$$

and denote the sum $\sum_{k \in Q(j)} x_k$ by $\sum_k x_k \ (k \in Q(j))$. A sequence x is said *to satisfy condition* $|Q|$ if $\sum_k |\nabla x_k| = o(1) \ (k \in Q(j))$.

We prove first the following theorem:

THEOREM 1. *Let A be a regular matrix. If $\sum \nabla x_k$ is a divergent series with bounded partial sums which is summed by A, then there exists an unbounded sequence $\{\lambda_k\}$ such that the series $\sum \lambda_k \nabla x_k$ is summed by A.*

Proof. Let x be the sequence of partial sums of $\sum \nabla x_k$. Notice first that it is sufficient to suppose that A sums x to zero, for if $\lim_A x = a \neq 0$, the sequence $\{x_k - a\}$ is summable A to zero and $\nabla x_k = \nabla(x_k - a)$ for $k > 1$. Now define $G = (g_{nk})$ by $g_{nk} = a_{nk} x_k$. Since x is bounded,

$$\sup_{n,r} \Big| \sum_{k=r}^{\infty} g_{nk} \Big| = \sup_{n,r} \Big| \sum_{k=r}^{\infty} a_{nk} x_k \Big| \leqslant \sup_n \sum |a_{nk}| \cdot |x_k| \leqslant \|A\| \cdot \|x\|.$$

Moreover, $a_{nk} \to 0$ as $n \to \infty$ and $|g_{nk}| = |a_{nk}| \cdot |x_k| \leqslant \|x\| \cdot |a_{nk}|$ and so $\lim_n g_{nk} = 0$. Finally,

$$\lim_n \sum_k g_{nk} = \lim_n \sum_k a_{nk} x_k = \lim_A x = 0.$$

Thus G is a conull, $v-c$ matrix and there exists an index sequence Q such that if y satisfies $|Q|$, then $y \in c_G$ ([3], p. 532, and [4], theorem 3.2). Also since x is a bounded divergent sequence there exists a sequence y such that

$$0 \leqslant \nabla y_n \leqslant |\nabla x_n|, \qquad y_n \uparrow \infty, \; y \text{ satisfies } |Q|.$$

Hence $\{x_k y_k\} \in c_A$.

Let $\{\lambda_k\}$ be any sequence such that $\lambda_k \nabla x_k = \nabla(x_k y_k)$. Since x is a bounded divergent sequence, $\nabla x_k \neq 0$ for infinitely many k, and for these k

$$\lambda_k = \nabla(x_k y_k)/\nabla x_k = y_k(\nabla x_k)/(\nabla x_k) + x_{k-1}(\nabla y_k)/(\nabla x_k)$$

and thus

$$|\lambda_k| \geqslant |y_k| - |x_{k-1}| \geqslant |y_k| - \|x\|.$$

Since $y_k \uparrow \infty$, the sequence $\{\lambda_k\}$ is unbounded. This completes the proof of the theorem.

Let x be an unbounded sequence and A be a regular matrix such that $c_A = \{ax + e\colon a \text{ is complex}, e \in c\}$ ([2], theorem 2). Suppose y is a divergent sequence and $\{\lambda_k\}$ is a sequence such that both ∇y and $\{\lambda_k \nabla y_k\}$ belong to ∇c_A. Then there exist complex numbers a, β ($a \neq 0$) and sequences $e, f \in c$ such that

$$\nabla y_k = a \nabla x_k + \nabla e_k, \qquad \lambda_k \nabla y_k = \beta \nabla x_k + \nabla f_k.$$

Thus, for k such that $\nabla y_k \neq 0$,

$$\begin{aligned} \lambda_k &= (\beta \nabla x_k + \nabla f_k)/(a \nabla x_k + \nabla e_k) \\ &= (\beta/a) + [a \nabla f_k - \beta \nabla e_k)/a] \cdot (a \nabla x_k + \nabla e_k)^{-1}. \end{aligned}$$

If x is chosen so that ∇x is bounded away from zero, then ∇y is bounded away from zero and, since $\nabla e, \nabla f \in c_0$, we have $\lim_k \lambda_k = \beta/a$. In particular, there exists a regular (normal) matrix A such that if A sums the divergent series $\sum_k \nabla y_k$ and A sums the series $\sum_k \lambda_k \nabla y_k$, then $\{\lambda_k\} \in c$.

We now give an application of theorem 1.

THEOREM 2. *Let A be a regular matrix which sums a bounded divergent sequence x, such that $\nabla x \notin c_0$. Then A sums a series with unbounded terms.*

Proof. Since x is a bounded divergent sequence summed by A, there exists, by theorem 1, an unbounded sequence $\{\lambda_k\}$ such that the series $\sum_k \lambda_k \nabla x_k$ is summed by A. We note, from the proof of theorem 1, that there is a y such that $y_k \uparrow \infty$ and $|\lambda_k| \geqslant |y_k| - \|x\|$ if $\nabla x_k \neq 0$ and that λ_k is arbitrary if $\nabla x_k = 0$. If, for k such that $\nabla x_k = 0$ we define $\lambda_k = y_k$, then $|\lambda_k| \geqslant |y_k| - \|x\|$ for each k and $\lim_k |\lambda_k| = \infty$. Since $\nabla x \notin c_0$, $\{\lambda_k \nabla x_k\}$ has an unbounded subsequence and the proof is complete.

An extension of theorem 2 and several related examples will appear elsewhere (1).

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(1) Some of these results were contained in the author's dissertation, written at Texas A. & M. University under the direction of Jack Bryant.