COLLOQUIUM MATHEMATICUM

VOL. XXIV 1971 FASC. 1

A NOTE ON POST ALGEBRAS

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Introduction. Over the last ten years Post algebras [8], [9], [19], have been studied from a number of points of view [6], [12], [4], [10], and in increasing generality [3], [16], [5], [10], [1]. In this note we present yet another point of view which, although introduced in [6] has not been exploited, and give another generalisation where the "constants" are allowed to form an arbitrary distributive lattice with zero and unit L.

We define a Post algebra $P = \langle B, L\rangle$ over a Boolean lattice B with lattice of constants L, and prove that P is lattice isomorphic to the lattice $\mathscr{C}(X, L)$ of all continuous L-valued functions defined on $X = \operatorname{Spec} B$ where L is equipped with the discrete topology. This representation gives a useful method of studying these more general Post algebras, and our results throw considerable light on the nature and limitations of many classical results.

As a further application we remark that the approach of this note appears to be useful in the study of Łukasiewicz algebras, see [18] and the references given there.

Notations. We shall always be working within the category $\mathbf{Dist}_{01}$ of distributive lattices with zero and unit; all sublattices have the same zero and unit as the over lattice; all morphisms preserve zero and unit. The complement of an element $a \in L$ will be denoted a^c. A partition of 1 in B is a subset $\{a_\alpha\} \subseteq B$ with $a_\alpha \wedge a_{\alpha'} = 0$ $(\alpha \neq \alpha')$ and $\bigvee_\alpha a_\alpha = 1$. If B is a Boolean lattice, the Stone space $X = \operatorname{Spec} B$ of B is the set of all maximal ideals x of B equipped with the topology generated by the basis sets $X_a = \{x \in X\colon a \notin x\}$ $(a \in B)$; the basic results on this topic can be found in [11]. Lattice congruences will always be denoted by upper case Greek letters with the corresponding lower case letters being used for the canonical epimorphism associated with the congruence.

1. Basic definitions. In this section we give our definition of a (generalised) Post algebra, and of its associated morphisms.

Definition 1.1. Let P be a distributive lattice and let B, L be a Boolean sublattice and a sublattice, respectively, of P. P is said to be the (generalised) *Post algebra over B with lattice L of constants* if:

(i) for any $s \in P$ there exists a finite partition $\{a_u\} \subseteq B$ of 1 and a set $\{l_u\} \subseteq L$ such that $s = \bigvee a_u \wedge l_u$;

(ii) the representation for $s \in P$ given in (i) is unique in the sense that if, for a finite partition $\{b_v\} \subseteq B$ of 1 and a set $\{m_v\} \subseteq L$ we also have $s = \bigvee b_v \wedge m_v$, then $\{b_v\}$ is a refinement of $\{a_u\}$ and $m_v = l_u$ whenever $b_v \leqslant a_u$.

We will call the representation for $s \in P$ given by (i) the *minimal representation* and write $P = \langle B, L\rangle$. It is straightforward to verify that when L is a finite chain and B is the centre of P, 1.1 reduces to the usual definition of a Post algebra; in fact our minimal representation is a slight variant of the monotonic representation. It is also clear that as a generalisation of the usual Post algebra, 1.1 differs from the ones previously given, e.g. [5], [16].

Definition 1.2. Let $P = \langle B, L\rangle$ and $P' = \langle B', L'\rangle$ be Post algebras. A lattice morphism $\varphi\colon P \to P'$ is called a *Post morphism* if $\varphi B \subseteq B$ and $\varphi L \subseteq L$.

2. Equivalent forms. The result of this section contains a generalisation of [10] and a clarification and extension of 3.2 of [1].

THEOREM 2.1. *Let P be a distributive lattice with Boolean sublattice B and sublattice L. Then the following are equivalent:*

(i) *$P = \langle B, L\rangle$, the Post algebra over B with constants L.*

(ii) *$P = B * L$, the coproduct of B and L in* $\mathbf{Dist}_{01}$.

(iii) *P is canonically isomorphic to $\mathscr{C}(X, L)$, the lattice of all L-valued functions on $X = \operatorname{Spec} B$, where L is equipped with the discrete topology.*

Proof. (i) $\Rightarrow$ (ii). We use the characterisation of a coproduct in $\mathbf{Dist}_{01}$ given in [1]. Suppose $a_1, a_2 \in B$, $l_1, l_2 \in L$ satisfy $a_1 \wedge l_1 \leqslant a_2 \vee l_2$. Writing this relation in the minimal representation of 1.1 (i) we have

$$(a_1 \wedge l_1) \vee (a_1^c \wedge 0)$$
$$= (a_1 \wedge a_2 \wedge l_1) \vee (a_1 \wedge a_2^c \wedge l_1 \wedge l_2) \vee (a_1^c \wedge a_2 \wedge 0 \wedge 1) \vee (a_1^c \wedge a_2^c \wedge 0 \wedge l_2)$$

whence, by the uniqueness 1.1 (ii), we deduce that $a_1 = a_1 \wedge a_2$ or $l_1 = l_1 \wedge l_2$ as required.

(ii) $\Rightarrow$ (iii). We will show that $\mathscr{C}(X, L)$ is the solution of the appropriate universal mapping problem. Firstly, if $a \in B$, $a \mapsto \bar{a} = \chi_{X_a}$ is a lattice monomorphism of B into $\mathscr{C}(X, L)$. Also if $\bar{l}\colon X \to L$ is the constant mapping to $l \in L$ then $l \to \bar{l}$ is a lattice monomorphism of L into $\mathscr{C}(X, L)$. Further, if $f \in \mathscr{C}(X, L)$ then the range of f is a finite subset say $\{l_u\} \subseteq L$. If $f^{-1} l_u = A_u \subseteq X$ then A_u is open-closed and so $A_u = X_{a_u}$ for a unique

$a_u \in B$. In this case $f = \bigvee_u \bar{a}_u \wedge \bar{l}_u$. Now if $\beta\colon B \to D$ and $\lambda\colon L \to D$ are lattice morphisms from B and L respectively into a distributive lattice D we can define $\gamma\colon \mathscr{C}(X, L) \to D$ by putting $\gamma f = \bigvee_u \beta a_u \wedge \lambda l_u$ where $f = \bigvee_u \bar{a}_u \wedge \bar{l}_u$. It is easy to see that γ is the unique morphism for which $\gamma a = \beta a$ $(a \in B)$, and $\gamma l = \lambda l$ $(l \in L)$.

(iii) $\Rightarrow$ (i). As remarked in the previous proof, any $f \in \mathscr{C}(X, L)$ can be written $f = \bigvee_u \bar{a}_u \wedge \bar{l}_u$ where it is easy to see that the $\{\bar{a}_u\} \subseteq \bar{B}$ so constructed is a partition of $\bar{1}$. Thus 1.1 (i) is satisfied and the construction we used clearly satisfies 1.1 (ii). The canonical isomorphism is $\bigvee_u a_u \wedge l_u$ (minimal representation) $\mapsto \bigvee_u \bar{a}_u \wedge \bar{l}_u$ where the latter is a function on X to L. This completes the proof of the theorem.

For the remainder of this note we shall take the canonical isomorphism of 2.1 to be an identification. Notationally, all this amounts to is the absence of a bar over elements of B and L.

3. Associated objects. We now turn to the definition and elementary properties of the objects associated with Post algebras, viz. Post subalgebras, Post ideals and Post congruences.

Let $P = \langle B, L\rangle$ and suppose that B_1 is a Boolean sublattice of B and L_1 a sublattice of L. We define the *Post subalgebra* generated by B_1, L_1 to be the sublattice P_1 of P generated by $B_1 \cup L_1$. Further, any sublattice obtained in this manner is called a Post subalgebra of P.

PROPOSITION 3.1. (i) *P_1 is the Post algebra $\langle B_1, L_1\rangle$.*

(ii) $P_1 = B_1 * L_1$.

(iii) *In the identification of 2.1, P_1 coincides with the sublattice of $\mathscr{C}(X, L)$ consisting of those functions which take values in L_1 and which can be factored through $X_1 = \operatorname{Spec} B_1$.*

Proof. We only prove (iii). First observe that the canonical monomorphism $i_1\colon B_1 \to B$ induces a continuous surjection $i_1^*\colon X \to X_1$ and so $f \in \mathscr{C}(X, L)$ factors through X_1 iff $f = f_1 \circ i_1^*$ where $f_1 \in \mathscr{C}(X_1, L)$. This gives a canonical isomorphism between the functions in (iii) and $\mathscr{C}(X_1, L_1)$ and the proof is complete.

Now suppose that j is an ideal of B and J an ideal of L where $P = \langle B, L\rangle$ is a Post algebra. We define the *Post ideal* $I_{j,J}$ generated by (j, J) to be the lattice ideal of P generated by $j \cup J$. Further, any ideal obtained in this manner is called a *Post ideal* of P.

PROPOSITION 3.2. *Let Q be a Post ideal of $P = \langle B, L\rangle$. Then:*

(i) *$Q = I_{j,J}$ where $j = Q \cap B$, $J = Q \cap L$.*

(ii) *In the identification of 2.1, Q coincides with the functions $\mathscr{I}_{j,J} = \{f \in \mathscr{C}(X, L)\colon f(x) \in J$ for all $x \supseteq j\}$.*

Proof. (i) is immediate. To see (ii) we first note that for $a \in B$, $\bar{a} = \chi_{X_a} \in \mathscr{I}_{j,J}$ iff $\bar{a}(x) = 0$ for all $x \supseteq j$, i.e. iff $a \in x$ for all $x \supseteq j$ which is clearly equivalent to $a \in j$. Also for $l \in L$, $\bar{l} \in \mathscr{I}_{j,J}$ iff $l \in J$. Thus $I_{j,J} \subseteq \mathscr{I}_{j,J}$, since the latter is clearly an ideal. The converse inclusion follows immediately from 1.1 (i).

A *Post congruence* Θ on a Post algebra $P = \langle B, L\rangle$ is a lattice congruence with the extra property: if $a, b \in B$ and $l, m \in L$, and $a \wedge l \equiv b \vee m(\Theta)$ then $a \wedge b \equiv a(\Theta)$ or $l \wedge m \equiv l(\Theta)$.

PROPOSITION 3.3. *Let Θ be a lattice congruence on the Post algebra $P = \langle B, L\rangle$ and let $\Phi = \Theta|B$ and $\Psi = \Theta|L$. The following are equivalent:*

(i) *Θ is a Post congruence.*

(ii) *Θ is a Post isomorphism of P/Θ onto $\langle B/\Phi, L/\Psi\rangle$.*

(iii) *For $s, t \in P$ we have $s \equiv t(\Theta)$ iff in $\mathscr{C}(X, L)$, $s(x) \equiv t(x)(\Psi)$ for all $x \supseteq \ker\Phi$.*

Proof. (i) and (ii) are clearly equivalent. Suppose Θ satisfies (iii); take $a, b \in B$ and $l, m \in L$ and suppose $a \wedge l \equiv b \vee m(\Theta)$ in P. Then if we assume $a \wedge b \not\equiv a(\Phi)$, there exists $x \supseteq \ker\Phi$ such that $b \in x$ and $a \notin x$. By (iii) we have $(\bar{a} \wedge \bar{l})(x) \equiv (\bar{b} \vee \bar{m})(x)(\Psi)$ in L, i.e. $\bar{a}(x) \wedge l \equiv \bar{b}(x) \vee m(\Psi)$. Since $a \notin x$, $\bar{a}(x) = 1$, and also $b \in x$ implies $\bar{b}(x) = 0$ whence $l \equiv m(\Psi)$ and Θ is a Post congruence.

To prove (i) implies (iii) we suppose Θ is a Post congruence; by (ii) P/Θ can be identified with $\mathscr{C}(X_\Phi, L/\Psi)$ where $X_\Phi = \operatorname{Spec} B/\Phi$. Now X_Φ is canonically homeomorphic with the closed subset $h(\ker\Phi) = \{x \in X: x \supseteq \ker\Phi\}$ of $X = \operatorname{Spec} B$. Consider the canonical epimorphism θ^*: $\mathscr{C}(X, L) \to \mathscr{C}(X_\Phi, L/\Psi) \approx \mathscr{C}\big(h(\ker\Phi), L/\Psi\big)$; the inverse image of $f \in \mathscr{C}\big(h(\ker\Phi), L/\Psi\big)$ under θ^* defines a congruence class exactly as described in (iii). The proof is complete.

Since congruences on B can be uniquely associated with ideals of B, a determination of all congruences on $P = \langle B, L\rangle$ is equivalent to a determination of ideals of B and congruences on L; we then apply 3.3 (iii). If J is an ideal of L and j an ideal of B we write $\Theta^{j,J}$ for the congruence on P obtained by 3.3 (iii), using the Φ on B determined by j and the Ψ on L determined by J.

4. Equivalence of two categories. It is clear from the preceding sections that the Boolean lattice B and the distributive lattice L independently determine the properties of $P = \langle B, L\rangle$. In this section we make this idea precise; to begin with we must reconstruct B and L from P.

PROPOSITION 4.1. *Let $P = \langle B, L\rangle$ and fix $x \in X$ and let p be a maximal ideal of L. Then the maps: λ: $L \to \mathscr{C}/\mathscr{I}_{x,0}$ given by $\lambda l = \theta^{x,0}(\bar{l})$ and β: $B \to \mathscr{C}/\mathscr{I}_{0,p}$, given by $\beta a = \theta^{0,p}(\bar{a})$, are lattice isomorphisms.*

Proof. We do not give full details. Consider first $\mathscr{C}/\mathscr{I}_{x,0}$; we observe that if $f \equiv g(\Theta^{x,0})$ then $f(x) = g(x)$. Also if $f(x) = l \in L$, say, we observe that $f \equiv \bar{l}(\Theta^{x,0})$, for if $A = f^{-1}(l)$, we have $f \vee \chi_{A^c} = \bar{l} \vee \chi_{A^c}$ where $\chi_{A^c}(x) = 0$. Thus congruence classes of $\mathscr{C}$ modulo $\mathscr{I}_{x,0}$ correspond in a one-one manner to elements of L. The remaining details that λ is an isomorphism are straightforward. Next consider $\mathscr{C}/\mathscr{I}_{0,p}$. If $f \equiv g(\Theta^{0,p})$ then it is not hard to show that, when $f = \bigvee_u \bar{a}_u \wedge \bar{l}_u$ and $g = \bigvee_v \bar{b}_v \wedge \bar{m}_v$ in the minimal representation, we have $\bigvee^B\{a_u : l_u \notin p\} = \bigvee^B\{b_v : m_v \notin p\}$. Conversely if for $h = \bigvee_w \bar{c}_w \wedge \bar{n}_w$ we have $\bigvee^B\{c_w : n_w \notin p\} = a \in B$, then we can show that $h \equiv \bar{a}(\Theta^{0,p})$.Thus β can be seen to be bijective and again the remaining details are omitted. This completes our outlined proof.

Our next result is in a sense an extension of Theorem 2.2 of [1], and is proved in the same way. **Post** is the category of (generalised) Post algebras introduced above, and **Bool** is the usual category of Boolean lattices.

THEOREM 4.2. *The categories* **Post** *and* **Bool** $\times$ **Dist**$_{01}$ *are equivalent.*

Proof. For the object $P = \langle B, L\rangle$ of **Post** we put $F(P) = (B, L)$; if $\varphi: P \to P'$ is a Post morphism we put $F(\varphi) = (\varphi|B, \varphi|L)$. It is easy to check that F is a functor from **Post** to **Bool** $\times$ **Dist**$_{01}$. Also for an object (B, L) of **Bool** $\times$ **Dist**$_{01}$ we put $G(B, L) = B * L$; if (φ_1, φ_2): $(B, L) \to (B', L')$ is a morphism in **Bool** $\times$ **Dist**$_{01}$ we define $G(\varphi)$ to be the unique morphism which follows from the definition of coproduct. Again G is easily checked to be a functor. The proof that F and G are mutually inverse category equivalences is exactly as in Theorem 2.2 of [1], making use of our 4.1. This completes our proof.

5. Further results. In this section we prove some miscellaneous results which relate to known results on (ordinary) Post algebras.

THEOREM 5.1. *The centre $Z_{\mathscr{C}}$ of $\mathscr{C}(X, L)$ coincides with the image of B iff $Z_L = \mathbf{2}$.*

Proof. If $Z_L = \mathbf{2}$ then any function f: $X \to L$ which has a complement f^c must take values in Z_L i.e. $f(x) = 0$ or 1 in L for all $x \in X$. Thus f is the indicator of its support and so in the image of B. Also if $a \in B$ then χ_{X_a} is central in $\mathscr{C}(X, L)$.

Conversely if $Z_{\mathscr{C}}$ coincides with the image of B in $\mathscr{C}(X, L)$ then $Z_L = \mathbf{2}$ since $n \in L$ with n^c must be $n = 0$ or $n = 1$. This completes the proof.

COROLLARY 5.2 ([1]). *If L is a chain, then the centre of $B * L$ is the image of B.*

Since we are dealing with a lattice of functions with values in L and pointwise operations, many properties of L carry over to $\mathscr{C}(X, L)$. Because this is of some interest we state some examples:

THEOREM 5.3. *Let $P = \mathscr{C}(X, L)$ be a Post algebra. If L is either a*

(i) *Pseudo-complemented lattice (or dually), or*

(ii) *Stone lattice* (*or dually*), *or*

(iii) *Relatively pseudo-complemented lattice* (*or dually*), *or*

(iv) *Post algebra*,

then so is P.

COROLLARY 5.4. *If* L *is dense* (*or dually*), *then* P *is a Stone lattice with centre isomorphic to* B (cf. [7]).

Next we fix the distributive lattice L and call Post algebras $P = \langle B, L\rangle$, Post L-algebras. Thus classical Post algebras are Post n-algebras where n is the n-element chain.

THEOREM 5.5. *The category* $\mathbf{Post}_L$ *of all Post* L*-algebras is closed under the operations of forming subalgebras, quotient algebras and products of algebras.*

Proof. We already have a notion of Post subalgebra and Post congruence (giving rise to quotient Post algebras) which specialise appropriately when L is fixed. It remains to check that $\mathbf{Post}_L$ is closed under products. Let $P_i = \langle B_i, L\rangle$ $(i \in I)$ be a family of Post L-algebras. Then we have $P_i \approx \mathscr{C}(X_i, L)$ $(i \in I)$ and so

$$P = \prod_{i \in I} P_i \approx \prod_{i \in I} \mathscr{C}(X_i, L) = \mathscr{C}(\dot{\bigcup_{i \in I}} X_i, L)$$

where $\dot{\bigcup}$ denotes disjoint union (topological coproduct); thus P is a Post L-algebra. It is clear that we can consider the action of L within a Post L-algebra as that of a family of operations.

COROLLARY 5.6 [15]. *Post* n*-algebras form a variety.*

Our representation of a Post algebra as a lattice of functions permits a very easy description of prime, minimal prime and maximal ideals.

THEOREM 5.7. *Let* $P = \mathscr{C}(X, L)$ *be a Post algebra. Then the ideal* $\mathscr{I}_{j,J} = \{f \in \mathscr{C}(X, L): f(y) \in J \text{ for all } y \supseteq j\}$ *is prime iff* $j = x \in X$ *and* J *is prime in* L. *Further the ideal* $\mathscr{I}_{x,p}$ *is minimal prime* [*maximal*] *iff* p *is minimal prime* [*maximal*].

Proof. We omit the easy details. Let us write Minp A, Max A respectively for the spaces of minimal prime and maximal ideals of an object A of $\mathbf{Dist}_{01}$. Then we have the easy

COROLLARY 5.8. $\operatorname{Minp}\mathscr{C}(X, L) \approx X \times \operatorname{Minp} L$; $\operatorname{Max}\mathscr{C}(X, L) \approx X \times \operatorname{Max} L$; $\operatorname{Spec}\mathscr{C}(X, L) \approx X \times \operatorname{Spec} L$.

From this corollary we can see how Dwinger [4] was led to define Post spaces, and that they are no more than products of Boolean spaces with spaces of the form $Y = \{1, 2, \ldots, n\}$ with topologies $\{\emptyset, \{1\}, \{1, 2\}, \ldots\}$.

6. Completeness. We close with a few remarks on completeness and completions of Post algebras. Although the results given undoubtedly hold for more general lattices L, in this preliminary note we only give them for finite L.

THEOREM 6.1. *Let L be finite. Then $\mathscr{C}(X, L)$ is $\mathfrak{m}$-complete ($\mathfrak{m}$ any infinite cardinal) iff B is $\mathfrak{m}$-complete.*

Proof. Take $\{f_i\colon i\in I\} \subseteq \mathscr{C}(X, L)$ with $|I| \leqslant \mathfrak{m}$. Then we show that $\bigvee_{i\in I} f_i$ can be defined pointwise and be continuous. This is true because for any $l\in L$, $\{x\in X\colon \bigvee_{i\in I} f_i(x) = l\} = \bigcup_{F\subseteq I} \{x\in X\colon \bigvee_{i\in F} f_i(x) = l\}$ where the union is over the finite subsets F of I; there are $\leqslant \mathfrak{m}$ such finite subsets and so if X is the representation space of an $\mathfrak{m}$-complete Boolean lattice the set union has open-closure, as each element is open-closed. Do this for all $l\in L$ and we see that $\bigvee_{i\in I} f_i$ is a continuous function on X. The converse is immediate since B itself is imbedded in $\mathscr{C}(X, L)$ and so known results [16] on Boolean lattices imply the desired result. The proof is complete.

The final result is now clear and can be stated without proof.

THEOREM 6.2. *Let L be finite and let $\bar{X} = \operatorname{Spec} \bar{B}$ where $\bar{B}$ is the normal completion of B. Then $\mathscr{C}(\bar{X}, L)$ is the normal completion of $\mathscr{C}(X, L)$.*

COROLLARY 6.3 [4]. *Put $L = n$ in 6.2.*

I would like to thank Dr. R. Balbes for sending the preprints [1], [2] thus stimulating me to write this note.

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Reçu par la Rédaction le 10. 6. 1970