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A MAPPING THEOREM FOR INFINITE-DIMENSIONAL MANIFOLDS AND ITS GENERALIZATIONS

BY

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0. Introduction. Paracompact (topological) manifolds modeled on a space E are called *E-manifolds*. We assume that all E-manifolds have the same weight as the model space E. As the model space we take a metrizable locally convex linear topological space E which is homeomorphic to ($\cong$) the countable-infinite product E^ω or its subspace

$$E_f^\omega = \{(x_i) \in E \mid x_i = 0 \text{ except for finitely many } i\}.$$

Let M be an E-manifold. A subset X of M is said to be *E-deficient* in M if there is a homeomorphism $h\colon M \to M \times E$ such that $h(X) \subset M \times \{0\}$. (Note that $M \cong M \times E$ by the Stability Theorem [25].) For a closed set X in M, X is E-deficient if any only if X is a *Z-set*, that is, for each open cover $\mathcal{U}$ of M there is a map $f\colon M \to M \setminus X$ which is $\mathcal{U}$-near to id (see [28], Remark B2). A *submanifold* of M is a subset of M which is an E-manifold. The following is well known as the Collaring Theorem (e.g., see [21], 4.4):

(A0) *Each closed submanifold of an E-manifold is E-deficient* (*i.e., a Z-set*) *if and only if it is collared* (*in the sense of Brown* [1]).

Such a submanifold is called a *Z-submanifold*. In view of the above fact, Z-submanifolds are considered as the abstract boundaries of E-manifolds. We will call a nowhere dense submanifold W of M a *boundary submanifold* if there exists an embedding $h\colon M \to E$ with $h(W) = \operatorname{bd} h(M)$, the topological boundary of $h(M)$ in E. In general, a Z-submanifold is not a boundary submanifold ([20], Example) and a boundary submanifold is not a Z-submanifold ([21], Example 2). It has been shown in [20] that a Z-submanifold W of M is a boundary submanifold if W contains some deformation retract of M.

M. Brown and B. Cassler [2] proved that each compact connected n-manifold M can be obtained from the n-cube I^n by making identifications on the boundary ∂I^n, that is, there is a map $h\colon I^n \to M$ such that $h(\partial I^n)$ is nowhere dense in M and $h|\mathring{I}^n$ is a homeomorphism of the interior $\mathring{I}^n$ onto $M\setminus h(\partial I^n)$. Prasad [19] established a similar result for Q-manifolds ($Q = I^\omega$, the Hilbert cube), that is, for each compact connected Q-manifold M there is a map h: $Q \times I \to M$ such that $h(Q \times \{1\})$ is nowhere dense in M and $h|Q \times [0, 1)$ is a homeomorphism of $Q \times [0, 1)$ onto $M \setminus h(Q \times \{1\})$. In this paper, we prove the E-manifold version, that is,

THEOREM I. *For any connected E-manifold M, there is a perfect map h*: $E \times I \to M$ *such that* $h(E \times \{1\})$ *is a boundary submanifold of M and* $h|E \times [0, 1)$ *is a homeomorphism of* $E \times [0, 1)$ *onto* $M \setminus h(E \times \{1\})$.

Here one should remark that $h(E \times \{1\})$ is a boundary submanifold of M. This is clearly impossible for finite-dimensional manifolds. Although Prasad's proof cannot conclude this, it is possible for Q-manifolds because our proof can be applied.

Theorem I is generalized as follows if the perfectness of h is not required:

THEOREM II. *Let* f: $M \to N$ *be a map between E-manifolds such that* $f(M)$ *meets all components of N and let* M_0 *be a Z-submanifold of M which is a deformation retract of M. Then f is homotopic to a map h*: $M \to N$ *such that* $h(M_0)$ *is a boundary submanifold of N and* $h|M \setminus M_0$ *is a homeomorphism of* $M \setminus M_0$ *onto* $N \setminus h(M_0)$.

To prove Theorem I, we show that each connected E-manifold is a perfect image of the model space E (Corollary 3.2). Then the following theorems are also generalizations of Theorem I, but unfortunately the proofs do not work for the following case:

(*) *E is not complete-metrizable and* $E \not\cong E_f^\omega$.

THEOREM III. *Excluding the case* (*), *let* f: $M \to N$ *be a surjective map between E-manifolds, and W a Z-submanifold of M such that M is deformable into W. Then f is homotopic to a map h*: $M \to N$ *such that* $h(W)$ *is a boundary submanifold of N and* $h|M \setminus W$ *is a homeomorphism of* $M \setminus W$ *onto* $N \setminus h(W)$ *and, for each* $x \in M$,

$$h^{-1}h(x) = x \quad \text{or} \quad h^{-1}h(x) \cong f^{-1}(y) \text{ for some } y \in N.$$

Moreover, if f is closed, then so is h. Thus, if f is perfect, light, monotone or UV^n $(1 \leqslant n \leqslant \infty)$, *then so is h.*

THEOREM IV. *Excluding the case* (*), *each surjective map* f: $M \to N$ *can be approximated by maps h*: $M \to N$ *such that, for some Z-submanifold* M_0 *which is a deformation retract of M,* $h(M_0)$ *is a boundary submanifold of N which is a deformation retract of N,* $h|M \setminus M_0$ *is a homeomorphism of* $M \setminus M_0$ *onto* $N \setminus h(M_0)$ *and* $h|M_0 = \psi f \phi^{-1}$ *for some homeomorphisms* ϕ: $M \to M_0$ *and* ψ: $N \to h(M_0)$. *Moreover, if f is closed, then so is h.*

All of our theorems are valid for compact Q-manifolds and also manifolds modeled on non-metrizable spaces,

$$\boldsymbol{R}^\infty = \operatorname{dir}\lim \boldsymbol{R}^n \quad \text{and} \quad Q^\infty = \operatorname{dir}\lim Q^n$$

(see Section 7).

1. Preliminaries. Let $\mathscr{U}$ be an open cover of a space Y. A map $f\colon X \to Y$ is *$\mathscr{U}$-near* to a map $g\colon X \to Y$ if for each $x \in X$ there is some $U \in \mathscr{U}$ such that $f(x), g(x) \in U$. A homotopy $h\colon X \times I \to Y$ is called a *$\mathscr{U}$-homotopy* if for each $x \in X$ there is some $U \in \mathscr{U}$ such that $h(\{x\} \times I) \subset U$. If h is a $\mathscr{U}$-homotopy from f to g (i.e., $h_0 = f$ and $h_1 = g$), we say that f is *$\mathscr{U}$-homotopic* to g. A map $f\colon X \to Y$ is a *$\mathscr{U}$-homotopy equivalence* if there is a map $g\colon Y \to X$ such that fg is $\mathscr{U}$-homotopic to id and gf is $f^{-1}(\mathscr{U})$-homotopic to id. A *fine homotopy equivalence* is a $\mathscr{U}$-homotopy equivalence for any open cover $\mathscr{U}$ of Y. A map $f\colon X \to Y$ is a *near homeomorphism* if for each open cover $\mathscr{U}$ of Y there is a homeomorphism $g\colon X \to Y$ which is $\mathscr{U}$-near to f. A closed map $f\colon X \to Y$ is *perfect* if each $f^{-1}(y)$ is compact.

Recall that E is a metrizable locally convex linear topological space such that $E \cong E^\omega$ or $\cong E^\omega_f$. By $\mathscr{E}$ we denote the class of all spaces which can be embedded in E as closed sets. Then $\mathscr{E}$ is a subclass of the class $\mathscr{M}$ of all metrizable spaces. By Dugundji's Extension Theorem ([12], Ch. II, Theorem 14.1), E is an $\mathrm{AE}(\mathscr{M})$; hence, by Hanner's Theorem ([12], Ch. II, Theorem 17.1), each E-manifold is an $\mathrm{ANE}(\mathscr{M})$, so an $\mathrm{ANE}(\mathscr{E})$. Recall we assume that E-manifolds have the same weight as E. By Henderson's result ([11], Theorem 2; cf. the proof of Theorem 1 in [10]), we have

(A1) *Each E-manifold can be embedded in E as a closed set.*

Hence

(A1′) *Each E-manifold is an* $\mathrm{ANR}(\mathscr{E})$.

For a complete-metrizable space X in the case (*) Toruńczyk [27] proved the following

(A2) *For each* $\mathrm{AR}(\mathscr{E})$ X, $X \times E \cong E$.

(A2′) *For each* $\mathrm{ANR}(\mathscr{E})$ X, *$X \times E$ is an E-manifold.*

Anderson–McCharen's Homeomorphism Extension Theorem was generalized in [5] (cf. [21]) combined with [28], Remark B2.

(A3) *Let M be an E-manifold and $\mathscr{U}$ an open cover of M. Each homeomorphism $f\colon X \to X'$ between Z-sets in M which is $\mathscr{U}$-homotopic to the inclusion $X \subset M$ can be extended to a homeomorphism $\tilde{f}\colon M \to M$ which is ambient-invertibly* $\mathrm{st}(\mathscr{U})$*-homotopic to* id.

Ferry [6] established the α-Approximation Theorem for l_2-manifolds. The proof is valid for E-manifolds because all results used in the proof have been generalized to E-manifolds. Thus we have

(A4) *Each $\mathscr{U}$-homotopy equivalence between E-manifolds is* $\mathrm{st}(\mathscr{U})$*-homotopic to a homeomorphism.*

This generalizes the Stability Theorem [25]:

(A4′) *For each E-manifold M, the projection p: $M \times E \to M$ is a near homeomorphism.*

For the following Triangulation Theorem, refer to [27], Theorem 3.4.

(A5) *Each E-manifold is homeomorphic to a product $|K| \times E$, where K is a locally finite-dimensional (l.f.d.) simplicial complex and $|K|$ admits the metric topology.*

2. The mapping cylinder of a map between ANR's. The *open cone* over a space X is a set

$$C^\circ(X) = X \times (0, \infty) \cup \{0\}$$

with the topology generated by open subsets of the product space $X \times (0, \infty)$ and sets $X \times (0, \varepsilon) \cup \{0\}$, $\varepsilon > 0$. The *mapping cylinder* of a map $f: X \to Y$ is a set

$$Z(f) = X \times (0, 1] \cup Y$$

with the topology generated by open subsets of the product space $X \times (0, 1]$ and sets $f^{-1}(V) \times (0, \varepsilon) \cup V$, where V is open in Y and $0 < \varepsilon < 1$. By $q: X \times I \to Z(f)$ we denote the *natural map* defined by

$$q(x, t) = \begin{cases} (x, t) & \text{if } t \neq 0, \\ f(x) & \text{if } t = 0. \end{cases}$$

The *collapsing* $c: Z(f) \to Y$ defined by

$$c(x, t) = f(x) \quad \text{for } (x, t) \in X \times (0, 1],$$
$$c(y) = y \quad \text{for } y \in Y$$

is continuous in this topology. When $Y = \{0\}$, $Z(f)$ is the *cone* over X, denoted by $C(X)$. Clearly,

$$C(X) \setminus X \times \{1\} \cong C^\circ(X).$$

In the above, *if X and Y are (complete-) metrizable, then the mapping cylinder Z(f) is also (complete-) metrizable.* By Kruse–Liebnitz's Theorem [14], *if X and Y are* ANR($\mathcal{M}$)*'s, then the mapping cylinder Z(f) is also an* ANR($\mathcal{M}$). Moreover, we have the following

2.1. Theorem. *Let $f: X \to Y$ be a map between* ANR($\mathcal{E}$)*'s. Then the mapping cylinder $Z(f)$ is also an* ANR($\mathcal{E}$). *And if Y is an* AR($\mathcal{E}$), *then so is $Z(f)$. Especially, the cone $C(X)$ over an* ANR($\mathcal{E}$) *X is an* AR($\mathcal{E}$).

This theorem is implied by the following

2.2. Lemma. *Let $f: X \to Y$ be a map. If X and Y can be embedded in E as closed sets (i.e., $X, Y \in \mathcal{E}$), then $Z(f)$ can be also embedded in E as a closed set.*

Proof. Assume that X and Y are closed sets in E. By Lemma 2 in [11] and its proof, $E \cong C^{\circ}(E)$, whence

$$E \cong E \times E \cong C^{\circ}(E) \times C^{\circ}(E).$$

Thus we may show that $Z(f)$ can be embedded in $C^{\circ}(E) \times C^{\circ}(E)$ as a closed set. Let $h: Z(f) \to C^{\circ}(E) \times C^{\circ}(E)$ be defined by

$$\begin{aligned} h(x, 1) &= ((x, 1), 0) && \text{for } (x, 1) \in X \times \{1\}, \\ h(x, t) &= ((x, t), (f(x), 1-t)) && \text{for } (x, t) \in X \times (0, 1), \\ h(y) &= (0, (y, 1)) && \text{for } y \in Y. \end{aligned}$$

Then it is straightforward to see that h is a closed embedding.

3. The proof of Theorem I. We first prove the following

3.1. LEMMA. *Let K be a connected l.f.d. simplicial complex. Then there are a contractible subcomplex L of the second barycentric subdivision* $\mathrm{sd}^2 K$ *and a perfect map g: $|L| \to |K|$ from $|L|$ onto $|K|$.*

Proof. Let T be the maximal tree of the 1-skeleton K^1 of K and let

$$L = \bigcup_{v \in K^0} \mathrm{st}(v, \mathrm{sd}^2 K) \cup \mathrm{sd}^2 T.$$

Then L is a contractible subcomplex of $\mathrm{sd}^2 K$. It is easy to construct a surjective (piecewise) linear map g: $|L| \to |K|$ such that, for each $v \in K^0$,

$$g\,\big||\mathrm{st}(v, \mathrm{sd}^2 K)|: |\mathrm{st}(v, \mathrm{sd}^2 K)| \to |\mathrm{st}(v, \mathrm{sd}\, K)|$$

is a homeomorphism and, for each $\sigma \in K^1 \setminus K^0$,

$$g(|\mathrm{st}(\hat{\sigma}, \mathrm{sd}^2 T)|) = \hat{\sigma},$$

where $\hat{\sigma}$ is the barycenter of σ. Observe that for each $x \in |K|$ there are only finitely many vertices $v \in K^0$ such that $x \in |\mathrm{st}(v, \mathrm{sd}\, K)|$ because K is l.f.d. Then it follows that g is perfect.

3.2. COROLLARY. *Each connected E-manifold M is a perfect image of the model space E.*

Proof. From (A5) we obtain $M \cong |K| \times E$ for some l.f.d. simplicial complex K. By the above lemma, we have a contractible subcomplex L of $\mathrm{sd}^2 K$ and a perfect map g: $|L| \to |K|$ of $|L|$ onto $|K|$. Since $|L| \times E \cong E$ by (A2), we have the result.

Proof of Theorem I. As seen in the above proof, there is a perfect map g: $|L| \to |K|$, where K and L are l.f.d. simplicial complexes such that

$$|K| \times E \cong M \quad \text{and} \quad |L| \times E \cong E.$$

Then the mapping cylinder $Z(g)$ is a complete-metrizable $\mathrm{ANR}(\mathscr{M})$ by 2.1, and hence $Z(g) \times E$ is an E-manifold by (A2′). Since the collapsing c: $Z(g)$

$\to |K|$ is a fine homotopy equivalence,

$$Z(g) \times E \cong |K| \times E \cong M$$

by (A4). As easily shown, $Z(g) \cup C(|K|)$ is a complete-metrizable AR($\mathscr{E}$), whence

$$(Z(g) \cup C(|K|)) \times E \cong E$$

by (A2). Observe that

$$\mathrm{bd}_{(Z(g) \cup C(|K|)) \times E}\, Z(g) \times E = |K| \times E.$$

Thus $|K| \times E$ is a boundary submanifold of $Z(g) \times E$. Since g is perfect, so is the natural map q: $|L| \times I \to Z(g)$. The restriction $q\,||L| \times (0, 1]$ is a homeomorphism onto the image

$$q(|L| \times (0, 1]) = Z(g) \setminus |K| = Z(g) \setminus q(|L| \times \{0\}).$$

By using (A3), we can show that

$$(|L| \times I \times E, |L| \times \{0\} \times E) \cong (E \times I, E \times \{1\}).$$

Then $q \times \mathrm{id}$: $|L| \times I \times E \to Z(g) \times E$ induces the required perfect map h: $E \times I \to M$.

4. The proof of Theorem II. Here we need the following

4.1. LEMMA. *Under the hypotheses of Theorem* II, *there are l.f.d. simplicial complexes K and L, a subcomplex K_0 of K, a surjective map g: $|K| \to |L|$ and homeomorphisms*

$$\phi: (|K| \times E, |K_0| \times E) \to (M, M_0) \quad \textit{and} \quad \psi: |L| \times E \to N$$

such that $\psi^{-1} f \phi$ is homotopic to $g \times \mathrm{id}$.

Proof. By Theorem 3.4 in [27] and Theorem 1.3 in [22], we have homeomorphisms

$$\phi_1: (|K_1| \times E, |K_0| \times E) \to (M, M_0) \quad \text{and} \quad \psi: |L| \times E \to N,$$

where K_1 and L are l.f.d. simplicial complexes and K_0 is a subcomplex of K_1. Then, obviously, $\psi^{-1} f \phi_1$ is homotopic to a map $g_1 \times \mathrm{id}$, where g_1: $|K_1| \to |L|$. By 3.1, there is a surjective (perfect) map g_2: $|K_2| \to |L|$, where K_2 is a contractible subcomplex of $\mathrm{sd}^2 L$. Then $|K_2| \times E \cong E$ by (A2). Take points $x_1 \in |K_1| \setminus |K_0|$ and $x_2 \in |K_2|$ so that $g_1(x_1) = g_2(x_2)$. By starring at x_1 and x_2, we can assume that $x_1 \in K_1^0$ and $x_2 \in K_2^0$. Identify $x_1 = x_2$ and let $K = K_1 \cup K_2$ be the one-point union ($K_1 \cap K_2 = \{x_1\} = \{x_2\}$). Then K is an l.f.d. simplicial complex and $|K|$ is complete-metrizable ANR($\mathscr{E}$). Let r: $|K| \to |K_1|$ be the retraction defined by $r(|K_2|) = \{x_1\}$ ($= \{x_2\}$). Since r is a fine homotopy equivalence, it is easy to construct a homeomorphism

$$\phi\colon (|K|\times E, |K_0|\times E) \to (M, M_0)$$

so that $\phi\colon |K|\times E \to M$ is homotopic to $\phi_1 \circ (r\times \mathrm{id})$ by using (A4) and (A3). We define a surjective map $g\colon |K| \to |L|$ by $g\,||K_i| = g_i$, $i = 1, 2$. Then g is homotopic to $g_1 r$. Hence $g\times\mathrm{id}$ is homotopic to $\psi^{-1} f\phi$.

Proof of Theorem II. By the above lemma, we can assume that $M = |K|\times E$, $M_0 = |K_0|\times E$ and $N = |L|\times E$, where K and L are l.f.d. simplicial complexes and K_0 is a subcomplex of K, and that $f = g\times\mathrm{id}$, where $g\colon |K| \to |L|$ is surjective. By using (A4) and (A3), we can easily construct a homeomorphism

$$j\colon (|K|\times I\times E, |K|\times\{0\}\times E) \to (|K|\times E, |K_0|\times E)$$

such that $j\colon |K|\times I\times E \to |K|\times E$ is homotopic to the projection. Then j^{-1} is homotopic to the embedding

$$i\colon |K|\times E \to |K|\times I\times E$$

defined by $i(x, y) = (x, 0, y)$. Using (A2′) and (A4), we have a homeomorphism $k\colon Z(g)\times E \to |L|\times E$ which is homotopic to $c\times\mathrm{id}$, where $c\colon Z(g) \to |L|$ is the collapsing. Define $h\colon |K|\times E \to |L|\times E$ as the composition

$$|K|\times E \xrightarrow{j^{-1}} |K|\times I\times E \xrightarrow{q\times\mathrm{id}} Z(g)\times E \xrightarrow{k} |L|\times E,$$

where $q\colon |K|\times I \to Z(g)$ is the natural map. Then h is homotopic to $cqi\times\mathrm{id} = g\times\mathrm{id} = f$. Similarly as in Theorem I, we can verify that h has the required property.

5. Boundary submanifolds. In this section, the case (*) is excluded. The following has been proved essentially in [20] by using (A1′), (A2), (A2′) and (A4):

5.1. PROPOSITION. *Let M be an E-manifold and W a closed nowhere dense submanifold of M such that*

$$(M, W) \cong (M\times E, W\times E).$$

If W contains a deformation retract of M, then W is a boundary submanifold of M.

Here is proved the next lemma:

5.2. LEMMA. *Let W be a Z-submanifold of an E-manifold M. If M is deformable into W, then the projection $p\colon M\times I \to M$ is homotopic to a homeomorphism $g\colon M\times I \to M$ such that $g(M\times\{0\}) \subset W$.*

Proof. Since any map between E-manifolds can be approximated by closed embeddings (see [10], p. 49, (a)) (this follows from (A1′) and (A4′)) and

M is deformable into W, we can easily obtain a closed embedding j: $M \to W \subset M$ which is homotopic to id. Then $j(M)$ is a Z-set in M because so is W. Let i: $M \to M \times I$ be the embedding defined by $i(x) = (x, 0)$. By (A2′) and (A4), the projection p: $M \times I \to M$ is a near homeomorphism, and hence homotopic to a homeomorphism k: $M \times I \to M$. Then $ki(M)$ is a Z-set in M because it is collared. Note that j and ki are homotopic. By (A3), we have a homeomorphism h: $M \to M$ such that $j = hki$ and h is ambient-invertibly isotopic to id. Then hk: $M \times I \to M$ is homotopic to p and

$$hk(M \times \{0\}) = hki(M) = j(M) \subset W.$$

By the above lemma, if an E-manifold M is deformable into a Z-submanifold W, then W contains a (strong) deformation retract of M. Thus the theorem of [20] is improved slightly:

5.3. Corollary. *Let W be a Z-submanifold of an E-manifold M. If M is deformable into W, then W is a boundary submanifold of M.*

5.4. Remark. Corollary 5.3 is also true for the case (*). In fact, Lemma 5.2 is valid for the case (*) and, moreover, we can assert that $g(M \times \{0\})$ is a Z-submanifold of W because the embedding j in the proof can be taken so that $j(M)$ is a Z-set in W. Thus W contains a Z-submanifold M_0 which is a (strong) deformation retract of M. Similarly as in [22], Theorem 1.3, we have a triple (K, L, K_0) of l.f.d. simplicial complexes such that

$$(M, W, M_0) \cong (|K| \times E, |L| \times E, |K_0| \times E).$$

Then $|K_0|$ is a deformation retract of $|K|$. By an easy modification of [20], we can prove that W is a boundary submanifold of M.

6. The proofs of Theorems III and IV. This section also excludes the case (*). We first prove the following lemma.

6.1. Lemma. *Let f: $M \to N$ be a surjective map between E-manifolds, and W a Z-submanifold of M such that M is deformable into W. Then the projection p: $M \times I \to M$ is homotopic to a homeomorphism g: $M \times I \to M$ such that $g(M \times \{0\}) \subset W$ and $qg^{-1}(W)$ is an* ANR$(\mathscr{M})$, *where q: $M \times I \to Z(f)$ is the natural map.*

Proof. By 5.2, p is homotopic to a homeomorphism g': $M \times I \to M$ such that $g'(M \times \{0\}) \subset W$. Since $M \times \{0\}$ is an ANR$(\mathscr{M})$ and a closed subset of an ANR$(\mathscr{M})$ $g'^{-1}(W)$, $M \times \{0\}$ is a strong neighborhood deformation retract of $g'^{-1}(W)$, that is, there are a neighborhood V of $M \times \{0\}$ in $g'^{-1}(W)$ and a homotopy h': $V \times I \to g'^{-1}(W)$ such that $h'_0 = \mathrm{id}$, $h'_1(V) = M \times \{0\}$ and $h'_t | M \times \{0\} = \mathrm{id}$ for each $t \in I$ (cf. [12], Ch. IV, Proposition 3.4). Let d and d' be metrics for M and N, respectively, such that

(1) $$d(x, x') \geqslant d'(f(x), f(x')) \quad \text{for each } x, x' \in M.$$

By $B(z, r)$ we denote the open ball in M or N with center z and radius r with respect to the metric d or d'. We can inductively construct maps v_n: $M \to (0, 2^{-n})$, $n \in N$, so that

$$(2) \qquad v_n(x) > v_{n+1}(x) \quad \text{for each } x \in M,$$

and for each $(x, s) \in g'^{-1}(W)$

$$(3) \qquad s < v_1(x) \text{ implies } (x, s) \in V,$$

$$(4) \qquad s < v_{n+1}(x) \text{ implies } h'(\{(x, s)\} \times I) \subset B(x, 2^{-n}) \times [0, v_n(x)).$$

By (2), we have a homeomorphism k: $M \times I \to M \times I$ isotopic to id such that

$$(5) \qquad pk = p,$$

$$(6) \qquad k|M \times \{0, 1\} = \text{id},$$

$$(7) \qquad k(x, 2^{-n}) = (x, v_n(x)) \quad \text{for each } x \in M \text{ and } n \in N.$$

Then we will show that $g = g'k$: $M \times I \to M$ is a desired homeomorphism. It is obvious that g is homotopic to p. By (6), we have $g(M \times \{0\}) \subset W$. Then it is easy to verify that $qg^{-1}(W)$ is closed in $Z(f)$ (cf. the proof of 2.1), and hence $qg^{-1}(W) \in \mathscr{E}$. To prove that $qg^{-1}(W)$ is an ANR$(\mathscr{E})$, we may show that it is an ANR$(\mathscr{M})$. From (5)–(7) and (3) it follows that

$$g'^{-1}(W) \cap k(M \times [0, 2^{-1})) \subset V.$$

Therefore

$$N \subset qk^{-1} g'^{-1}(W) \cap (M \times (0, 2^{-1}) \cup N) \subset qk^{-1}(V);$$

hence $qk^{-1}(V)$ is a neighborhood of N in $qg^{-1}(W) = qk^{-1} g'^{-1}(W)$. Now we define

$$h\colon qk^{-1}(V) \times I \to qg^{-1}(W) = qk^{-1} g'^{-1}(W)$$

as follows:

$$h((x, s), t) = qk^{-1} h'(k(x, s), t) \quad \text{for } (x, s) \in qk^{-1}(V) \cap M \times (0, 1],$$
$$h(y, t) = y \quad \text{for } y \in N.$$

From (4)–(7) and (1) it follows that

$$h(f^{-1}(B(y, 2^{-n-1})) \times (0, 2^{-n-1}) \times I)$$
$$\subset f^{-1}(B(y, 2^{-n-1})) \times (0, 2^{-n}) \cup B(y, 2^{-n}).$$

This implies the continuity of h. Since h is a homotopy such that $h_0 = \text{id}$, $h_1(qk^{-1}(V)) = N$ and $h_t|N = \text{id}$ for each $t \in I$, N is a strong neighborhood retract of $qg^{-1}(W) = qk^{-1} g'^{-1}(W)$. Note that

$$qg^{-1}(W) \setminus N = g^{-1}(W) \setminus M \times \{0\}$$

is an ANR($\mathscr{M}$) because it is open in an ANR($\mathscr{M}$) $g^{-1}(W) \cong W$. By Kruse–Liebnitz's Theorem [14], $qg^{-1}(W)$ is an ANR($\mathscr{M}$).

Proof of Theorem III. By the above lemma, we have a homeomorphism $g\colon M \times I \to M$ homotopic to the projection such that $g(M \times \{0\}) \subset W$ and $qg^{-1}(W)$ is an ANR($\mathscr{E}$). Since W is a Z-submanifold of M, we can easily construct a homeomorphism

$$\theta\colon (M \times E,\ W \times E) \to (M,\ W)$$

homotopic to the projection by using (A4′) and (A3) (cf. [21], Section 4). Then

$$(g^{-1} \times \mathrm{id}) \circ \theta^{-1}\colon M \to M \times I \times E$$

is homotopic to the embedding $i\colon M \to M \times I \times E$ defined by $i(x) = (x, 0, 0)$ because $\theta \circ (g \times \mathrm{id}) \circ i$ is homotopic to id. By 2.1, $Z(f)$ is an ANR($\mathscr{E}$), and hence $Z(f) \times E$ is an E-manifold by (A2′). Let $\pi\colon Z(f) \times E \to Z(f)$ be the projection, and $c\colon Z(f) \to N$ the collapsing. Obviously, $c\pi\colon Z(f) \times E \to N$ is a fine homotopy equivalence, and so a near homeomorphism by (A4). Then we have a homeomorphism $k\colon Z(f) \times E \to N$ which is homotopic to πc. Now we will show that the composition

$$M \xrightarrow{\theta^{-1}} M \times E \xrightarrow{g^{-1} \times \mathrm{id}} M \times I \times E \xrightarrow{q \times \mathrm{id}} Z(f) \times E \xrightarrow{k} N$$

is the desired map $h\colon M \to N$. In fact, h is homotopic to $c\pi \circ (q \times \mathrm{id}) \circ i = f$ and $f(X) = Y$ implies $q(M \times I) = Z(f)$, so $h(M) = N$. Since N is a (strong) deformation retract of $Z(f)$, $k(N \times E)$ is so in $k\big(Z(f) \times E\big) = N$. Observe that

$$h(W) = k\big(qg^{-1}(W) \times E\big) \supset k\big(q(M \times \{0\}) \times E\big) = k(N \times E).$$

Hence $h(W)$ contains a deformation retract of N. Since $qg^{-1}(W)$ is an ANR($\mathscr{E}$), $qg^{-1}(W) \times E$ is an E-manifold by (A2′), so $h(W)$ is a submanifold of N. And, moreover,

$$\begin{aligned}(N,\ h(W)) &\cong \big(Z(f) \times E,\ qg^{-1}(W) \times E\big)\\ &\cong \big(Z(f) \times E \times E,\ qg^{-1}(W) \times E \times E\big) \cong \big(N \times E,\ h(W) \times E\big).\end{aligned}$$

It is easy to see that $qg^{-1}(W)$ is nowhere dense in $Z(f)$, and hence $h(W)$ is nowhere dense in N. Therefore, by 5.1, $h(W)$ is a boundary submanifold of N. Since

$$q|M \times (0, 1]\colon M \times (0, 1] \to Z(f) \setminus N$$

is a homeomorphism, $h|M \setminus W\colon M \setminus W \to N \setminus h(W)$ is a homeomorphism. The additional statements are easily verified.

Proof of Theorem IV. In the above proof, the homeomorphism $k: Z(f)\times E \to N$ can be $\mathscr{V}$-near to $c\pi$ for a given open cover $\mathscr{V}$ of N. The projection $p'\colon M\times I\times E\to M$ is a near homeomorphism, whence $f^{-1}(\mathscr{V})$-near to a homeomorphism $g'\colon M\times I\times E\to M$. For each $x\in M$ there is some $V\in\mathscr{V}$ such that

$$x(=g'g'^{-1}(x)),\ p'g'^{-1}(x)\in f^{-1}(V).$$

Therefore

$$i(x),\ g'^{-1}(x)\in f^{-1}(V)\times I\times E=(q\times\mathrm{id})^{-1}\pi^{-1}c^{-1}(V),$$

whence

$$f(x)(=c\pi\circ(q\times\mathrm{id})\circ i(x)),\ c\pi\circ(q\times\mathrm{id})\circ g'^{-1}(x)\in V.$$

This implies that f is $\mathscr{V}$-near to $c\pi\circ(q\times\mathrm{id})\circ g'^{-1}$, so $\mathrm{st}(\mathscr{V})$-near to $k\circ(q\times\mathrm{id})\circ g'^{-1}$. Then $M_0=g'(M\times\{0\}\times E)$ is a boundary Z-submanifold of M and

$$k\circ(q\times\mathrm{id})\circ g'^{-1}(M_0)=k(N\times E)$$

is a boundary submanifold of N. Thus we have the result.

7. Non-metrizable infinite-dimensional manifolds and Q-manifolds. Our arguments in this paper are also valid for manifolds modeled on

$$\boldsymbol{R}^\infty=\operatorname{dir}\lim\boldsymbol{R}^n \quad\text{and}\quad Q^\infty=\operatorname{dir}\lim Q^n,$$

that is, $\boldsymbol{R}^\infty$-manifolds and Q^∞-manifolds. However, some modifications are necessary.

As seen in [16], Section 4, we cannot use Z-sets as a characterization of $\boldsymbol{R}^\infty$- or Q^∞-deficient closed sets in $\boldsymbol{R}^\infty$- or Q^∞-manifolds. The author [24] introduced D-sets which characterize $\boldsymbol{R}^\infty$- or Q^∞-deficient closed sets. We have need to replace the words "Z-set" and "Z-submanifold" by "D-set" and "D-submanifold", respectively. The $\boldsymbol{R}^\infty$- and Q^∞-versions of (A0) are due to [18], Theorems 3.3 and 3.4 (cf. [24], 7.3).

By $\mathscr{DC}^{\mathrm{fd}}$ and $\mathscr{DC}$ we denote the classes of all (countable) direct limits of finite-dimensional compact metrizable spaces and of all (countable) direct limits of compact metrizable spaces. By [24], 4.1, the classes $\mathscr{DC}^{\mathrm{fd}}$ and $\mathscr{DC}$ are the classes of all spaces which can be embedded in $\boldsymbol{R}^\infty$ and Q^∞ as closed sets, respectively. The $\boldsymbol{R}^\infty$- and Q^∞-versions of (A1) are Theorem II.2 (b) or Proposition III.2 in [7]. By Dugundji's Extension Theorem (see [12], Ch. II, Theorem 14.1, and [13], Theorem 10.1), $\boldsymbol{R}^\infty$ and Q^∞ are $\mathrm{AE}(\mathscr{M}^{\mathrm{H}})$'s, where $\mathscr{M}^{\mathrm{H}}$ is the class of all M-spaces introduced by Hyman [13]. Note that $\mathscr{DC}\cup\mathscr{M}\subset\mathscr{M}^{\mathrm{H}}$. Spaces in $\mathscr{DC}^{\mathrm{fd}}$ or $\mathscr{DC}$ are $\mathrm{ANR}(\mathscr{DC}^{\mathrm{fd}})$'s or $\mathrm{ANR}(\mathscr{DC})$'s if and only if they are ANR($\mathscr{M}^{\mathrm{H}}$)'s. The $\boldsymbol{R}^{\infty}$- and Q^{∞}-versions of (A1′) are obtained by Hanner's Theorem ([12], Ch. II, Theorem 17.1; cf. [7], Corollary II.4).

In the $\boldsymbol{R}^{\infty}$- and Q^{∞}-cases, we use the mapping cylinder with the usual quotient topology. By Z_f we denote the mapping cylinder of a map $f: X \to Y$ with the quotient topology. It is not difficult to see that if X and Y belong to $\mathscr{D}\mathscr{C}^{\mathrm{fd}}$ or $\mathscr{D}\mathscr{C}$, then so does Z_f. By Hyman's Theorem ([13], Theorem 11.1), if X and Y are ANR($\mathscr{D}\mathscr{C}^{\mathrm{fd}}$)'s or ANR($\mathscr{D}\mathscr{C}$)'s, then so is Z_f.

Using the author's characterizations of $\boldsymbol{R}^{\infty}$- and Q^{∞}-manifolds ([23], Theorem 1.3), we can easily obtain the versions of (A2) and (A2′). The versions of (A3) and (A4) have been established by Liem [15]–[18] (see also [23] and [24]). The versions of (A5) are proved in [8] and [9] (cf. [23]). Then proofs in Sections 3–5 apply to $\boldsymbol{R}^{\infty}$- and Q^{∞}-manifolds. Thus we can obtain the same results for $\boldsymbol{R}^{\infty}$- and Q^{∞}-manifolds.

Our results are also valid for *compact* Q-manifolds because all statements in Section 1 are true ((A4) is a little different statement; cf. [6]). In proofs, some changes might be needed but easy. For Q-manifolds, refer to [3].

There is no difference between $(Q \times [0, 1))$-manifolds and Q-manifolds ([3], Theorem 12.1). However, in general, $M \times Q \times [0, 1) \ncong M$ for a Q-manifold M. For a Q-manifold M, $M \times Q \times [0, 1) \cong M$ if and only if $M \times [0, 1) \cong M$. Such a Q-manifold is said to be $[0, 1)$-*stable*. Our results are valid for $E = Q \times [0, 1)$ and $[0, 1)$-stable Q-manifolds because all statements in Section 1 are true.

Acknowledgment. In the first manuscript, the author established the $\boldsymbol{R}^{\infty}$-version of Brown's mapping theorem by an inductive method, and then proved the σ-version by some modifications with metric arguments. The Q^{∞}- and Σ-versions were obtained as corollaries to $\boldsymbol{R}^{\infty}$- and σ-versions. (Here σ and Σ are the subspaces of the separable Hilbert space l_2 which are linear spans of the usual orthonormal basis and the Hilbert cube $\prod_{n \in \boldsymbol{N}} [-2^{-n}, 2^{-n}] \subset l_2$, respectively). By the f-d cap set arguments in [4], the l_2-version was proved as a completion of the σ-version. However, this original proof cannot apply to more general (non-separable) spaces. When the author visited the Banach International Mathematical Center at Warsaw for the Topology Semester 1984, H. Toruńczyk suggested to him that a mapping cylinder is available for the proof of the infinite-dimensional version of Brown's mapping theorem (Theorem I). By using mapping cylinders, it was succeeded to generalize Theorem I to Theorems II, III and IV. In (A2) and (A2′) we assume that X is complete-metrizable in the case $(*)$, but it is unknown whether this assumption is necessary or not. Thus we do not know whether

Theorems III and IV are valid in the case (*). This was also pointed out by Toruńczyk. The author wishes to express his sincere thanks to Henryk Toruńczyk for his kind and valuable suggestions.

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