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ON THE JOIN OF EQUATIONAL CLASSES OF IDEMPOTENT ALGEBRAS AND ALGEBRAS WITH CONSTANTS

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We will consider algebras of a given type $\tau = (n_1, n_2, \ldots)$ with corresponding fundamental polynomials $f_1(x_1, \ldots, x_{n_1}), f_2(x_1, \ldots, x_{n_2}), \ldots$, where $n_k > 0$ for $k = 1, 2, 3, \ldots$

We say that an algebra $\mathfrak{A}$ is *idempotent* if $f_k(x, \ldots, x) = x$ for $k = 1, 2, 3, \ldots$

Let $\mathscr{K}_1$ be an equational class of idempotent algebras and $\mathscr{K}_2$ be an equational class of algebras in which any polynomial f_k is a constant, i.e. $f_k(x_1, \ldots, x_{n_k}) = f_k(y_1, \ldots, y_{n_k})$. Let $\mathscr{K}_1 \vee \mathscr{K}_2$ be the join of $\mathscr{K}_1$ and $\mathscr{K}_2$, i.e. the smallest equational class containing $\mathscr{K}_1$ and $\mathscr{K}_2$. In this paper we describe algebras from $\mathscr{K}_1 \vee \mathscr{K}_2$ by means of algebras from $\mathscr{K}_1$ and $\mathscr{K}_2$.

First we define a construction of algebras.

Let $\mathfrak{A}_0 = \langle I; f_1, f_2, \ldots \rangle$ be an idempotent algebra and let $\{\mathfrak{A}_i \mid \mathfrak{A}_i = \langle X_i; f_1, f_2, \ldots \rangle, i \in I\}$ be a family of algebras in which all f_k are constants. Let us denote the constant operation $f_k(x_1, \ldots, x_{n_k})$ in an algebra $\mathfrak{A}_i$ by $k(i)$.

Now we define a new algebra of the same type τ, namely $\mathop{S}\limits_{\mathfrak{A}_0} \mathfrak{A}_i$, as

$$\mathop{S}\limits_{\mathfrak{A}_0} \mathfrak{A}_i = \langle \bigcup_{i \in I} X_i; f_1, f_2, \ldots \rangle,$$

where $\bigcup\limits_{i \in I} X_i$ is the disjoint union of the sets X_i, and if $a_1 \in X_{i_1}, a_2 \in X_{i_2}, a_3 \in X_{i_3}, \ldots, a_{n_k} \in X_{i_{n_k}}$, then $f_k(a_1, \ldots, a_{n_k}) = k\big(f_k(i_1, i_2, \ldots, i_{n_k})\big)$, where $f_k(i_1, \ldots, i_{n_k})$ is taken in the algebra $\mathfrak{A}_0$.

THEOREM 1. *Let $\mathfrak{A} = \langle X; f_1, f_2, \ldots \rangle$ be an algebra of type τ. Then the following three conditions are equivalent:*

(C1) *$\mathfrak{A} \in \mathscr{K}_1 \vee \mathscr{K}_2$, where $\mathscr{K}_1$ is an equational class of idempotent algebras, and $\mathscr{K}_2$ is an equational class of algebras in which any polynomial f_k is a constant.*

(C2) $\mathfrak{A}$ *satisfies all identities of the form*

$$(1)\quad f_k(x_1, \ldots, x_{n_k}) = f_k\big(f_k(x_1, \ldots, x_{n_k}), \ldots, f_k(x_1, \ldots, x_{n_k})\big) = f_k\big(f_l(x_1, \ldots, x_1), f_l(x_2, \ldots, x_2), \ldots, f_l(x_{n_k}, \ldots, x_{n_k})\big).$$

(C3) $\mathfrak{A} = \sum_{\mathfrak{A}_0} \mathfrak{A}_i$, *where* $\mathfrak{A}_0$ *is an idempotent algebra and in any algebra* $\mathfrak{A}_i$ *any polynomial* f_k *is a constant.*

Proof. The implication (C1)$\Rightarrow$(C2) follows from the fact that (1) is satisfied in $\mathscr{K}_1$ because of idempotency, and also in $\mathscr{K}_2$ because in algebras from $\mathscr{K}_2$ any f_k is a constant. Thus (1) is satisfied in $\mathscr{K}_1 \vee \mathscr{K}_2$.

(C2)$\Rightarrow$(C3). Let us define a relation R in X by putting $aRb \Leftrightarrow f_1(a, \ldots, a) = f_1(b, \ldots, b)$. The relation R is obviously an equivalence. We shall show that it is a congruence in $\mathfrak{A}$. Let $a_j R b_j$, $j = 1, 2, \ldots, n_k$. Thus, by (1),

$$\begin{aligned} f_1\big(f_k(a_1, \ldots, a_{n_k}), f_k(a_1, \ldots, a_{n_k}), \ldots, f_k(a_1, \ldots, a_{n_k})\big) &= f_1\Big(f_k\big(f_1(a_1, \ldots, a_1), f_1(a_2, \ldots, a_2), \ldots, f_1(a_{n_k}, \ldots, a_{n_k})\big), \ldots, \\ &\qquad f_k\big(f_1(a_1, \ldots, a_1), f_1(a_2, \ldots, a_2), \ldots, f_1(a_{n_k}, \ldots, a_{n_k})\big)\Big) \\ &= f_1\Big(f_k\big(f_1(b_1, \ldots, b_1), f_1(b_2, \ldots, b_2), \ldots, f_1(b_{n_k}, \ldots, b_{n_k})\big), \ldots, \\ &\qquad f_k\big(f_1(b_1, \ldots, b_1), f_1(b_2, \ldots, b_2), \ldots, f_1(b_{n_k}, \ldots, b_{n_k})\big)\Big) \\ &= f_1\big(f_k(b_1, \ldots, b_{n_k}), f_k(b_1, \ldots, b_{n_k}), \ldots, f_k(b_1, \ldots, b_{n_k})\big). \end{aligned}$$

Thus R is a congruence. By (1) we have

$$f_1(x, \ldots, x) = f_1\big(f_k(x, \ldots, x), \ldots, f_k(x, \ldots, x)\big),$$

whence $xRf_k(x, \ldots, x)$, what means that $\mathfrak{A}/R$ is an idempotent algebra.

Write $\mathfrak{A}/R = \mathfrak{A}_0 = \langle I; f_1, f_2, \ldots \rangle$, where I is the set of indices of congruence classes of the relation R. We shall show that any congruence class X_i is a subalgebra of $\mathfrak{A}$, and any function f_k is a constant in $\mathfrak{A}_i$. Let $a_1, a_2, \ldots, a_{n_k}, b \in X_i$. Then, by (1), we have

$$\begin{aligned} & f_1\big(f_k(a_1, \ldots, a_{n_k}), \ldots, f_k(a_1, \ldots, a_{n_k})\big) \\ &= f_1\Big(f_k\big(f_1(a_1, \ldots, a_1), f_1(a_2, \ldots, a_2), \ldots, f_1(a_{n_k}, \ldots, a_{n_k})\big), \ldots, \\ &\qquad f_k\big(f_1(a_1, \ldots, a_1), f_1(a_2, \ldots, a_2), \ldots, f_1(a_{n_k}, \ldots, a_{n_k})\big)\Big) \\ &= f_1\Big(f_k\big(f_1(b, \ldots, b), \ldots, f_1(b, \ldots, b)\big), \ldots, f_k\big(f_1(b, \ldots, b), \ldots, f_1(b, \ldots, b)\big)\Big) \\ &= f_1\big(f_k(b, \ldots, b), \ldots, f_k(b, \ldots, b)\big) = f_1(b, \ldots, b). \end{aligned}$$

Thus $f_k(a_1, \ldots, a_{n_k})Rb$ and X_i is a subalgebra.

Further,

$$\begin{aligned} f_k(a_1, \ldots, a_{n_k}) &= f_k\big(f_1(a_1, \ldots, a_1), f_1(a_2, \ldots, a_2), \ldots, f_1(a_{n_k}, \ldots, a_{n_k})\big) \\ &= f_k\big(f_1(b, \ldots, b), \ldots, f_1(b, \ldots, b)\big) = f_k(b, \ldots, b), \end{aligned}$$

and so f_k is a constant in X_i. Denote this constant by $k(i)$.

Let $a_1 \in X_{i_1}, a_2 \in X_{i_2}, \ldots, a_{n_k} \in X_{i_{n_k}}$. Since R is a congruence relation, we have $f_k(a_1, \ldots, a_{n_k}) \in X_{f_k(i_1, \ldots, i_{n_k})}$, but

$$f_k(a_1, \ldots, a_{n_k}) = f_k\big(f_k(a_1, \ldots, a_{n_k}), \ldots, f_k(a_1, \ldots, a_{n_k})\big) = k\big(f_k(i_1, \ldots, i_{n_k})\big).$$

(C3)$\Rightarrow$(C1). Let $\mathscr{K}_1$ be the equational class generated by $\mathfrak{A}_0$, and $\mathscr{K}_2$ — the equational class generated by all algebras $\mathfrak{A}_i$. Let $\varphi = \psi$ be an identity in $\mathscr{K}_1 \vee \mathscr{K}_2$. Then it follows from the definition of $\underset{\mathfrak{A}_0}{S} \mathfrak{A}_i$ that $\underset{\mathfrak{A}_0}{S} \mathfrak{A}_i \in \mathscr{K}_1 \vee \mathscr{K}_2$.

In fact, if we substitute arguments in the identity $\varphi = \psi$, then the value of φ and ψ must be in the same X_i, and then $\varphi = \psi$ for an arbitrary choice of arguments if and only if the most external polynomials in φ and ψ are equal in $\mathscr{K}_2$.

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