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REMARKS ON THE APPLICATION OF RAMSEY NUMBERS TO BOUNDING THE SUM OF DENSITIES OF A GRAPH AND ITS COMPLEMENT

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All graphs in this note are simple, that is contain no loops and multiple edges. The cardinality of a maximum-sized independent set of vertices in G is denoted by $\alpha(G)$ and the cardinality of the maximum-sized complete subgraph of G is denoted by $\omega(G)$ and called the *density* of G. Note that $\alpha(G) = \omega(\bar{G})$, where $\bar{G}$ is the complement graph of G.

In the paper [1], it was proved (Theorem 2) that in the class $\mathscr{P}$ of graphs whose components are complete graphs $K_{m_1}, K_{m_2}, \ldots, K_{m_k}$ and which is closed with respect to taking a complement (that is, $K_{m_1,m_2,\ldots,m_k} \in \mathscr{P}$), we have

$$(1)\qquad S(n) = \min(\alpha(G)+\alpha(\bar{G})) = \min(\alpha(G)+\omega(G)) = \min(\omega(G)+\omega(\bar{G})) = 2[\sqrt{n}]+\varepsilon_n,$$

where $\varepsilon_n = 0$ for $q = \sqrt{n}-[\sqrt{n}] = 0$, $\varepsilon_n = 1$ for $q \leqslant \frac{1}{2}$ and $\varepsilon_n = 2$ for $q > \frac{1}{2}$, $n = |G| = |\bar{G}| = m_1+m_2+\ldots+m_k$. There, we posed also the problem of determining all graphs for which this formula holds.

It is easy to see that (1) does not hold in the class $\mathscr{G}$ of all simple graphs. Looking for bounds in other subclasses of $\mathscr{G}$ (closed with respect to taking a complement) we observed that this bound is connected with Ramsey numbers related to a given class.

Let $\mathscr{H}$ be a subclass of $\mathscr{G}$ closed with respect to taking a complement. The Ramsey number $r(K, \mathscr{H}, n_1, n_2)$ denotes the minimum integer such that if $\omega(G) < n_1$ then $\omega(\bar{G}) \geqslant n_2$ for every $G \in \mathscr{H}$, which has at least $r(K, \mathscr{H}, n_1, n_2)$ vertices. We assume here that $n_1 \leqslant n_2$.

In the present note we restrict attention to the class $\mathscr{P}$ and relate the calculation of $S(n)$ to Ramsey numbers $r(K, \mathscr{P}, n_1, n_2)$. Hence we obtain a different proof of Theorem 2 from [1]. Let $p(n_1, n_2)$ denote $r(K, \mathscr{P}, n_1, n_2)$. We first prove the following theorem:

THEOREM 1. *There holds*

$$p(n_1, n_2) = (n_1-1)(n_2-1)+1.$$

Proof. Let $G \in \mathscr{P}$, $\omega(G) \leqslant n_1 - 1$, $|G| > (n_1-1)(n_2-1)$, and let $\bar{G} = K_{m_1, m_2, \ldots, m_q}$. Then the number q of components of G is equal to at least n_2, thus for the complement $\bar{G}$ we have $\omega(\bar{G}) \geqslant n_2$. If $G = K_{m_1, m_2, \ldots, m_q}$, then from the assumption $\omega(G) \leqslant n_1 - 1$ it follows that the size of a largest component in $\bar{G}$ is equal to at least $\omega(\bar{G}) \geqslant n_2$. If $|G| = (n_1-1)(n_2-1)$ then, assuming that $\omega(G) = n_1 - 1$, we obtain $\bar{G}$ for which $q = n_2 - 1$, and $m_1 = m_2 = \ldots = m_q = (n_1-1)$ and $|G| = |\bar{G}| = (n_1-1)(n_2-1)$. Hence, taking into account that for every n there exists $G \in \mathscr{P}$ such that $|G| = n$, the thesis of the theorem follows.

Theorem 1 results in a crude bound to $r(K, \mathscr{H}, n_1, n_2)$ for any class $\mathscr{H} \supset \mathscr{P}$, in particular we have $r(K, \mathscr{G}, n_1, n_2) \geqslant (n_1-1)(n_2-1)+1$.

In the class $\mathscr{P}$, by the obvious inequality $(n_1-k-1)(n_1+k-1)+1 \leqslant (n_1-1)^2+1$ we have the following bound:

LEMMA 1. *There holds* $p(n_1-k, n_1+k) \leqslant p(n_1, n_1)$ *for every* $k = 1, 2, \ldots, n_1-2$.

Analogously, by the inequality $(n_1-k-1)(n_1+k)+1 \leqslant (n_1-1)n_1+1$ which holds for every $k \geqslant 0$ we have the following bound:

LEMMA 2. *There holds* $p(n_1-k, n_1+k+1) \leqslant p(n_1, n_1+1)$ *for every* $k = 1, 2, \ldots, n_1-2$.

If $G \in \mathscr{P}$ and $|G| \geqslant p(n_1, n_1)$, then from the definition of p, $\omega(G) < n_1$ implies $\omega(\bar{G}) \geqslant n_1$. From the existence of the graph G_0 such that $\omega(G_0) = n_1 - 1$ and $\omega(\bar{G}_0) = n_1$ we have $\omega(G_0)+\omega(\bar{G}_0) = 2n_1 - 1$. By Lemma 1, if $\omega(G) = n_1 - k$ then $\omega(\bar{G}) \geqslant n_1 - k$ for $k = 1, 2, \ldots, n_1-2$. Using now the obvious inequality $(n_1-k)+(n_1+k) \geqslant (n_1-1)+n_1$, we obtain $\min(\omega(G)+\omega(\bar{G})) = 2n_1 - 1$.

Analogously, we can prove that if $G \in \mathscr{P}$ and $|G| \geqslant p(n_1, n_1+1)$ then we have $\min(\omega(G)+\omega(\bar{G})) = 2n_1$.

Thus we have proved

THEOREM 2. *Let* $G \in \mathscr{P}$. *If* $n = |G| \in (p(n_1, n_1), p(n_1, n_1+1))$ *then* $S(n) = \min(\omega(G)+\omega(\bar{G})) = 2n_1 - 1$ *and if* $n \in (p(n_1, n_1+1), p(n_1+1, n_1+1))$ *then* $S(n) = 2n_1$.

Probably, Lemmas 1 and 2 are satisfied for at least every subclass $\mathscr{H}$ of $\mathscr{G}$ such that (similarly as in $\mathscr{P}$ or $\mathscr{G}$) for every n and for every $k = 1, 2, \ldots, n$ there exists a graph $G \in \mathscr{H}$ such that $|G| = n$ and $\omega(G) = k$. If for a given class $\mathscr{H}$ one could derive an explicit formula for $r(K, \mathscr{H}, n_1, n_2)$ as in Theorem 1, such a formula would allow then to prove counterparts of Lemmas 1 and 2, and Theorem 2.

A somewhat different situation may arise in a class $\mathscr{H}$ which does not satisfy for every n the condition that there exist graphs with sufficiently large density. As an example may serve the class $\mathcal{S}$ of self-complementary graphs, i.e. $G \in \mathcal{S}$ if and only if G is isomorphic with $\bar{G}$. Such graphs exist only for $n = |G|$ or $n-1$ divisible by 4. From the condition that a graph and its complement are isomorphic we get only Ramsey numbers of the form $r(K, \mathcal{S}, m, m)$. Let us denote $r(K, \mathcal{S}, m, m)$ simply by $s(m)$. Hence, $s(m)$ is the minimum $|G|$ such that every $G \in \mathcal{S}$, $|G| \geqslant s(m)$, contains K_m. In this class of graphs, $S(n)$ assumes only even values. The values of $r(K, \mathcal{S}, m, m)$ imply $s(3) = 8$, $s(4) = 20$ and $S(4) = S(5) = 4$, $S(8) = S(9) = S(12) = S(13) = S(16) = S(17) = 6$, $S(20) = S(21) = 8$. For $n > 12$, this is only a conjecture based on the hypothesis that among the graphs in $\mathcal{G}$ for which an even minimum value of $S(n)$ is achieved there exist self-complementary graphs for the appropriate values of n and that for odd values of $S(n)$ there exist self-complementary graphs G such that $\omega(G)+\omega(\bar{G}) = 2\omega(G) = S(n)+1$.

The above considerations lead to an interesting problem of investigating the subclass $\mathcal{G}_0 \subset G$ in which the value of $S(n)$ is minimum. Examples of such graphs allow to conjecture that they cannot have a small (or large) number of edges. This guess appears however to be false if one considers examples of disconnected graphs in $\mathcal{G}_0$. Let us consider $G = K_2 \cup K_2 \cup K_2$ for $n = 6$ which has three edges. For $n = 9$ we have $G = C_5 \cup K_2 \cup K_2$ with 7 edges, and for $n = 10$ we have $G = C_5 \cup C_5$ with 10 edges. The number of edges increases here along with n, but the increase is very slow. The investigation of the subclass of graphs which belong to $\mathcal{G}_0$ and which for a given n have the smallest number of edges seems to be also interesting. For $|G| = n = 4$ we have three graphs: $K_2 \cup K_2$, P_3 and C_4 which belong to $\mathcal{G}_0$; P_3 is self-complementary. For $n = 5$ we have only one graph C_5 with $S(5) = 4$. This graph is also self-complementary and we have $2\omega(C_5) = 6 = \max(\omega(C_5)+\omega(\bar{C}_5))$. For $n = 6$ we know already five disconnected graphs which belong to $\mathcal{G}_0$: $K_2 \cup K_2 \cup K_2$, $C_3 \cup C_3$, $C_5 \cup K_1$, $C_4 \cup K_2$, $P_3 \cup K_2$ as well as several connected graphs. It is interesting to note that for $n = 7$ we have only one disconnected graph $C_5 \cup K_2$ in $\mathcal{G}_0$ and that for $n = 8$ there exists no such graph.

The existence of at least one Hamiltonian graph in $\mathcal{G}_0$ for each n seems to be quite certain. A more difficult problem presents an answer to the question whether for $n > 4$ there exists a self-complementary Hamiltonian graph with minimum value of $S(n)$ (**P 1300**). The construction of such graphs for $n = 5$, 8, 9, 12 presents no difficulty.

REFERENCE

[1] L. Szamkołowicz, *Sur la classification des graphes en vue des propriétés de leurs noyaux*, Prace Naukowe Instytutu Matematyki i Fizyki Teoretycznej Politechniki Wrocławskiej 3, 1970, p. 15–21.

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