COLLOQUIUM MATHEMATICUM

VOL. XXX 1974 FASC. 1

m-EXISTENTIALLY COMPLETE STRUCTURES

BY

D. SARACINO (NEW HAVEN, CONNECTICUT)

1. Introduction. The main purpose of this paper* is to show how A. Robinson's generic structures for infinite forcing within a class Σ of structures can be obtained using standard model-theoretic techniques (as indicated in [3], but with some simplifications).

It has been observed by various people that there are strong similarities between the behaviour of the class of generic structures and the class of existentially complete structures, with regard to such notions as ultraproducts and elementary classes. For example, one is an EC_Δ iff the other is, in which case they coincide (see, for example, [2]). We will present two different iterations of the notion of existential completeness, either of which yields a characterization of the generic structures in case Σ is closed under unions of chains. In fact, if Σ is inductive, the two iterations coincide; for general Σ, they may differ and reflect to different degrees the behaviour of the generic structures.

For convenience, we recall a few basic definitions and facts from Robinson's paper [1]. Let Σ be a class of structures of some fixed similarity type. If $A \in \Sigma$ and φ has constants from A (i.e., φ is defined in A), the relation $A \Vdash \varphi$ (A *forces* φ) is defined by induction on the structure of formulas as follows:

(i) If φ is atomic, $A \Vdash \varphi$ iff $A \models \varphi$.

(ii) If φ is $\psi \wedge \chi$, $A \Vdash \varphi$ iff $A \Vdash \psi$ and $A \Vdash \chi$.

(iii) If φ is $\psi \vee \chi$, $A \Vdash \varphi$ iff $A \Vdash \psi$ or $A \Vdash \chi$ (or both).

(iv) If φ is $\exists x \psi(x)$, $A \Vdash \varphi$ iff, for some constant c denoting an element of A, $A \Vdash \psi(c)$.

(v) If φ is $\neg\psi$, $A \Vdash \varphi$ iff for no extension B of A in Σ is it the case that $B \Vdash \psi$.

A structure A in Σ is said to be *generic* if, for any φ defined in A, $A \models \varphi$ iff $A \Vdash \varphi$. It is easy to see that if A and B are both generic and

* Research supported by National Science Foundation Graduate Fellowship.

$A \subset B$, then $A \prec B$, and, furthermore, that any generic structure is *existentially complete* in Σ, i.e. any existential sentence defined in Σ which holds in some extension of A in Σ already holds in A. Robinson shows in [1] that if Σ is inductive, then every A in Σ has an extension B in Σ which is generic.

By an $\exists_0$ (or $\forall_0$) *formula* we mean a quantifier-free formula. Assuming that $\exists_n$ and $\forall_n$ are defined for $n < m$, we define $\exists_m$ as the union of $\forall_{m-1}$ and the set of all formulas of the form $\exists x_1 \ldots \exists x_r \psi$, where $\psi \in \forall_{m-1}$, and $\forall_m$ as the union of $\exists_{m-1}$ and the set of all formulas of the form $\forall x_1 \ldots \forall x_s \psi$, where $\psi \in \exists_{m-1}$. If A and B are structures, we write $A \prec_n B$ to mean that $A \subset B$ and that, for every formula $\psi \in \forall_n$ with constants from A, $A \models \psi$ iff $B \models \psi$.

We establish the convention that all structures are in Σ; thus, when we say "there exists an extension B of A...", we mean "there exists an extension B of A in Σ...".

2. m-existentially complete structures.

Definition. Any structure is 0-existentially complete (0-e.c.). Assuming that "n-existentially complete" is defined for $n < m$, we say that A is *m-existentially complete* if, for any extension B of A, there exists an extension C of B which is $(m-1)$-e.c. and such that $A \prec_m C$.

PROPOSITION 1. *If A is m-e.c., then A is n-e.c. for any $n < m$.*

Proof. Clearly, any 1-e.c. structure is 0-e.c. Assuming the result for $r < m$, let A be m-e.c. Then, for any $B \supset A$, there exists a $C \supset B$ which is $(m-1)$-e.c. and such that $A \prec_m C$. By induction hypothesis, C is $(m-2)$-e.c., and, clearly, $A \prec_{m-1} C$. So A is $(m-1)$-e.c., and hence, by induction hypothesis, n-e.c. for all $n < m$.

PROPOSITION 2. *Suppose $m \geqslant 1$. If A is m-e.c., B is $(m-1)$-e.c. and $A \subset B$, then $A \prec_m B$.*

Proof. If A is 1-e.c. and $B \supset A$, then there exists a $C \supset B$ such that $A \prec_1 C$. Then, clearly, $A \prec_1 B$. Assuming the result for $n < m$, let A be m-e.c., and B an $(m-1)$-e.c. extension of A. Let C be an $(m-1)$-e.c. extension of B such that $A \prec_m C$. Suppose $\psi \in \forall_m$ is defined in A, $A \models \psi$ and B non $\models \psi$. Then $B \models \neg\psi$, and $\neg\psi$ is equivalent to an element of $\exists_m$. By Proposition 1, C is $(m-2)$-e.c., so, by induction hypothesis, $B \prec_{m-1} C$; therefore, $C \models \neg\psi$. But $C \models \psi$ since $A \prec_m C$.

For the rest of this section we assume that Σ is inductive.

The connection between generic and m-e.c. structures is the next result.

THEOREM 1. *For any A, A is generic iff A is m-e.c. for all m.*

Proof. Suppose, first, that A is generic. We prove that A is m-e.c. for all m by induction on m. For $m = 1$ the result is clear because any generic structure is existentially complete and any existentially complete structure is obviously 1-e.c. Assuming the result for all $n < m$, suppose that A is generic and let B be any extension of A. Let (by inductivity) C be a generic extension of B. Then, by induction hypothesis, C is $(m-1)$-e.c., and, clearly, $A \prec_m C$ because, in fact, $A \prec C$. So A is m-e.c., and the induction is complete.

For the converse, suppose that A is m-e.c. for all M and let B be any generic extension of A. Then, by the first part of the proof, B is m-e.c. for all m. Therefore, by using the prenex normal form and Proposition 2, we can conclude that $A \prec B$. We have shown that A is an elementary substructure of any generic structure extending it. Therefore, A is generic: an easy induction on formulas shows that it suffices to consider $\neg\psi$ defined in A such that $A \models \neg\psi$ and prove $A \Vdash \neg\psi$. So suppose A non $\Vdash \neg\psi$. Then some extension B of A forces ψ; if C is (by inductivity of Σ) a generic extension of B, then $C \Vdash \psi$, so $C \models \psi$. But $A \prec C$, so $C \models \neg\psi$, a contradiction.

Remark. We have just shown that if $A \subset B$ and A and B are both m-e.c. for all m, then $A \prec B$. This is the analogue of the fact that $A \subset B$ implies $A \prec B$ for A and B generic, in the forcing-theoretic set-up.

It immediately seems reasonable that we should be able to take "m-e.c. for all m" as our starting point and thus, for example, obtain forcing companions (cf. [1]) without using any forcing. The main obstacle is, of course, that we would have to prove that the structures m-e.c. for all m are cofinal in Σ. We will do so without using forcing, but first we give a result that elucidates the relationship between forcing and being m-e.c. for a particular m.

THEOREM 2. *Fix m. Then A is m-e.c. iff, for any φ in $\forall_m$ defined in A, $A \models \varphi$ iff $A \Vdash \varphi$.*

Proof. For $m = 0$, the result is trivial. Assume it proved for all $n < m$. Suppose A is m-e.c. and $\varphi \in \forall_m$ is defined in A. Say

$$\varphi = \forall x_1 \ldots \forall x_r \psi(x_1, \ldots, x_r) \quad \text{with } \psi \text{ in } \exists_{m-1}.$$

Suppose $A \models \varphi$. To show that $A \Vdash \varphi$ we must show that, for any b_1 in any $B_1 \supset A_1$, some extension C_1 of B_1 forces $\forall x_2 \ldots \forall x_r \psi(b_1, x_2, \ldots, x_r)$. Take for C_1 an $(m-1)$-e.c. extension of B_1 such that $A \prec_m C$. To prove that C_1 does the job we must show that, for any b_2 in any $B_2 \supset C_1$, some $C_2 \supset B_2$ forces $\forall x_3 \ldots \forall x_r \psi(b_1, b_2, x_3, \ldots, x_r)$. Take for C_2 an $(m-1)$-e.c. extension of B_2 such that $A \prec_m C_2$. Continuing in this way we arrive at

$$A \subset B_1 \subset C_1 \subset B_2 \subset C_2 \subset B_3 \subset C_3 \subset \ldots \subset B_r \subset C_r$$

$$(b_i \in B_i \text{ for } i = 1, 2, \ldots, r),$$

where C_r is an $(m-1)$-e.c. extension of B_r such that $A \prec_m C_r$, and we must show that $C_r \Vdash \psi(b_1, \ldots, b_r)$. Now $C_r \models \varphi$ since $\varphi \in \forall_m$ is defined in A, so, in particular, $C_r \models \psi(b_1, \ldots, b_r)$. Since, by induction hypothesis, forcing and satisfaction coincide in C_r for statements in $\forall_{m-1}$, it is clear that they coincide for statements in $\exists_{m-1}$. Therefore, $C_r \Vdash \psi(b_1, \ldots, b_r)$.

Now suppose $A \Vdash \varphi$. To show that $A \models \varphi$ we must prove that, for any $b_1, \ldots, b_r$ in A, $A \models \psi(b_1, \ldots, b_r)$. Certainly, some extension B of A forces $\psi(b_1, \ldots, b_r)$, so, letting C be an $(m-1)$-e.c. extension of B such that $A \prec_m C$, we have $C \Vdash \psi(b_1, \ldots, b_r)$, so since forcing and satisfaction coincide in C for sentences in $\exists_{m-1}$, $C \models \psi(b_1, \ldots, b_r)$. Since $A \prec_m C$, $A \models \psi(b_1, \ldots, b_r)$, as desired.

For the converse of the theorem, suppose that, for any φ in $\forall_m$ defined in A, $A \models \varphi$ iff $A \Vdash \varphi$. Let B be any extension of A. Let C be a generic extension of B. Then, supposing that an element φ of $\forall_m$ is defined in A and $A \models \varphi$, we have $A \Vdash \varphi$ whence $C \Vdash \varphi$ and so $C \models \varphi$. Hence $A \prec_m C$ and the proof is complete.

Remark. When we introduced C in the last section, we used the cofinality in Σ of the generic structures, so our proof as it stands uses a result that depends on a forcing argument. Once we have proved the cofinality of the m-e.c. structures we can take for C any m-e.c. extension of B and use the first half of the theorem to conclude $C \models \varphi$, thus removing the dependence on a forcing argument.

PROPOSITION 3. *For any m, the union of an increasing chain of m-e.c. structures is itself m-e.c.*

Proof. For $m = 0$, this is clear. Assume the result for all $n < m$. Suppose our chain is $\{A_\mu\}_{\mu<\lambda}$, and B is an extension of $A = \bigcup_{\mu<\lambda} A_\lambda$. Let C be an $(m-1)$-e.c. extension of B such that $A_0 \prec_m C$. Suppose that φ is in $\forall_m$ and is defined in A, and that $A \models \varphi$. Then φ is defined in some A_ξ; since, by induction hypothesis, A is $(m-1)$-e.c., Proposition 2 yields $A_\xi \prec_m A$, so $A_\xi \models \varphi$. But, also by Proposition 2, $A_\xi \prec_m C$, so $C \models \varphi$. We have shown that $A \prec_m C$, so A is m-e.c.

Remark. Thus the class of structures m-e.c. for all m is inductive. This is our counterpart of Robinson's result (see [1]) that the class of generic structures is inductive.

THEOREM 3. *For any m, the class of m-e.c. structures is cofinal in Σ.*

Proof. For $m = 0$, this is trivial. Assume that, for any $n < m$, the class of n-e.c. structures is cofinal in Σ.

Given A in Σ, let A^0 be an $(m-1)$-e.c. extension of A. We first construct an $(m-1)$-e.c. extension A^1 of A^0 such that any φ in $\exists_m$ which is defined in A^0 and holds in some $(m-1)$-e.c. extension of A^1 holds already in A^1. To do this let $\{\varphi_\mu\}_{\mu<\lambda}$ be a listing of all the formulas in $\exists_m$ with constants in A^0. If φ_0 holds in some $(m-1)$-e.c. extension of A^0, let A_0^0 be such an extension; otherwise, let $A_0^0 = A^0$.

Now suppose $\mu < \lambda$ and assume that A_ξ^0 has been defined for all $\xi < \mu$. If μ is a successor ordinal $\mu = \xi+1$ and φ_μ holds in some $(m-1)$-e.c. extension of A_ξ^0, let A_μ^0 be such an extension; otherwise, let $A_\mu^0 = A_\xi^0$. If μ is a limit ordinal and φ_μ holds in some $(m-1)$-e.c. extension of $\bigcup_{\xi<\mu} A_\xi^0$, let A_μ^0 be such an extension; otherwise, let

$$A_\mu^0 = \bigcup_{\xi<\mu} A_\xi^0.$$

Let us assume that

$$A^1 = \bigcup_{\mu<\lambda} A_\mu^0.$$

Now suppose some φ_μ holds in some $(m-1)$-e.c. extension of A^1. Then, if $\mu = \xi+1$, φ_μ holds in some $(m-1)$-e.c. extension of A_ξ^0, and if $\mu = \bigcup\mu$, then φ_μ holds in some $(m-1)$-e.c. extension of $\bigcup_{\xi<\mu} A_\xi^0$. Therefore, by construction, $A_\mu^0 \models \varphi_\mu$. An easy induction using Proposition 3 shows that each A_μ^0 is $(m-1)$-e.c., so, by Proposition 2, $A^1 \models \varphi_\mu$. Therefore, A^1 has the desired properties.

Now, let A^2 be an $(m-1)$-e.c. extension of A^1 obtained from A^1 just as A^1 was obtained from A^0; in this way form a chain $A^0 \subset A^1 \subset A^2 \subset \ldots$ and let B be the union of this chain. We claim that B is m-e.c.

Suppose that φ is in $\exists_m$ and defined in B, and that φ holds in some $(m-1)$-e.c. extension of B. Then φ is defined in some A^j, and φ now holds in some $(m-1)$-e.c. extension of A^{j+1}, so, by construction, $A^{j+1} \models \varphi$. Since B is $(m-1)$-e.c. by Proposition 3, Proposition 2 yields $B \models \varphi$.

Let C be any extension of B and let D be any $(m-1)$-e.c. extension of C. Then, by the previous argument, $B \prec_m D$, so B is m-e.c. This completes the proof.

COROLLARY. *The structures "m-e.c. for all m" are cofinal in Σ.*

Proof. Given A, let A^1 be a 1-e.c. extension of A; let A^2 be a 2-e.c. extension of A^1; let A^3 be a 3-e.c. extension of A^2; and so on. Let B be the union of the chain so formed. For any m, all the A^j for $j \geqslant m$ are m-e.c., so B is m-e.c. by Proposition 3.

An interesting question regarding the classes of m-e.c. structures is that of when the sequence degenerates, i.e. stops at some point m. An easy sufficient condition is given by the following

THEOREM 4. *Let Σ^F denote the set of all sentences in the language appropriate for Σ which hold in all generic structures in Σ. Suppose that, for any predicate $\varphi(x_1, \ldots, x_r)$, there is a predicate ψ_φ in $\exists_m$ such that*

$$\Sigma^F \vdash \forall x_1 \ldots \forall x_r(\varphi(x_1, \ldots, x_r) \leftrightarrow \psi_\varphi(x_1, \ldots, x_r)).$$

Then any m-e.c. structure is generic.

Proof. Let A be m-e.c. and let B be a generic extension of A. Suppose $B \models \exists x\varphi(x, a_1, \ldots, a_r)$ with a_i in A. Then, for some b in B, $B \models \varphi(b, a_1, \ldots, a_r)$, so, by hypothesis, $B \models \psi_\varphi(b, a_1, \ldots, a_r)$ with ψ_φ in $\exists_m$. Therefore, $B \models \exists x\psi_\varphi(x, a_1, \ldots, a_r)$, and since $A \prec_m B$, we have $A \models \exists x\psi_\varphi(x, a_1, \ldots, a_r)$. So, for some a in A, $A \models \psi_\varphi(a, a_1, \ldots, a_r)$; therefore, $B \models \psi_\varphi(a, a_1, \ldots, a_r)$, so $B \models \varphi(a, a_1, \ldots, a_r)$. By the Tarski-Vaught test, $A \prec B$. Therefore, A is an elementary subsystem of any of its generic extensions, so it is generic, as in the proof of Theorem 1.

3. Weakly m-e.c. structures. Another reasonable iteration of the notion of existential completeness can be given as follows. (We drop the assumption that Σ is inductive, unless otherwise noted.)

Definition. Any structure is weakly 0-e.c.; assuming that the class of weakly m-e.c. structures has been defined for all $n < m$, we say that a *structure is weakly m-e.c.* if it is weakly $(m-1)$-e.c. and has the property that, for any weakly $(m-1)$-e.c. extension B, $A \prec_m B$.

Notice that, for any class Σ, both the 1-e.c. structures and the weakly 1-e.c. structures are precisely the existentially complete structures.

We will show that, for an inductive Σ, "m-e.c." and "weakly m-e.c." coincide for all m. However, if Σ is not assumed inductive, then any m-e.c. structure is weakly m-e.c., but the converse may fail. Also, for any Σ, not necessarily inductive, an m-e.c. structure A has the property that, for any ψ in $\forall_m$ defined in it, $A \models \psi$ iff $A \Vdash \psi$; this may fail for a weakly m-e.c. structure A.

THEOREM 5. (i) *For an arbitrary class Σ, not assumed inductive, and for any m, any m-e.c. structure is weakly m-e.c.*

(ii) *For Σ inductive and for any m, a structure is m-e.c. iff it is weakly m-e.c.*

Proof. (i) By induction. For $m = 0$, the result is trivial; assume it proved for $n < m$. Let A be m-e.c.; then A is $(m-1)$-e.c., so, by induction hypothesis, A is weakly $(m-1)$-e.c. Let B be any weakly $(m-1)$-e.c. extension of A; we must show that $A \prec_m B$. Let C be an $(m-1)$-e.c. extension of B such that $A \prec_m C$. By induction hypothesis, C is weakly $(m-1)$-e.c., so, by the definition of the weakly m-e.c. structures and by the fact that B is weakly $(m-1)$-e.c., $B \prec_{m-1} C$. We have $B \prec_{m-1} C$ and $A \prec_m C$; so $A \prec_m B$. This completes the induction.

(ii) By (i), it suffices to show that any weakly m-e.c. structure is m-e.c. So, let B be any extension of a given weakly m-e.c. structure A; by Theorem 3, let C be an $(m-1)$-e.c. extension of B. By part (i), C is weakly $(m-1)$-e.c., so since A is weakly m-e.c., $A \prec_m C$. Therefore, A is m-e.c.

THEOREM 6. *Let Σ be arbitrary and A an m-e.c. structure. Then, for any ψ in $\forall_m$ defined in A, $A \models \psi$ iff $A \Vdash \psi$.*

Proof. All we have to do is to examine the "only if" part of the proof of Theorem 2 to see that we did not use the inductivity of Σ to prove it.

Example. Let A_0 be an algebraically closed field of characteristic zero. Let $A_0 \subset A_1 \subset A_2 \subset \ldots$ be a chain of extensions of A_0 such that

(i) for any $i > 0$, there is an irreducible polynomial over A_i which has a root in some A_j for $j > i$ but no root in A_i, and

(ii) for any $i > 0$, there is an irreducible polynomial over A_i which has no root in any A_j.

Let Σ consist of the A_i, $0 \leqslant i < \omega$.

It is clear, by condition (i), that no A_i for $i > 0$ is existentially complete in Σ. Since A_0 is existentially complete in Σ, it follows that A_0 is weakly m-e.c. for all m. However, A_0 is not m-e.c. for all m; in fact, it is not even 2-e.c., because, for example, A_1 is an extension of it which has no 1-e.c. extension C such that $A_0 \prec_2 C$ (A_1 has no 1-e.c. extensions at all).

We also note that although A_0 is weakly m-e.c. for all m, there exist $\forall_2$-sentences ψ such that $A_0 \models \psi$ but A_0 does not force ψ. For let $a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with coefficients in A_1 which has no root in any A_j. Then $A_0 \models \psi$, where

$$\psi = \forall y_0 \ldots \forall y_n \exists x (y_n x^n + \ldots + y_1 x + y_0 = 0),$$

but if $A_0 \Vdash \psi$, then, in particular, some extension A_j of A_1 forces

$$\exists x (a_n x^n + \ldots + a_1 x + a_0 = 0),$$

so

$$A_j \models \exists x (a_n x^n + \ldots + a_1 x + a_0 = 0),$$

a contradiction.

In particular, of course, A_0 is not generic.

In closing we remark that results somewhat similar to ours have been obtained independently by G. Cherlin (as yet unpublished). Cherlin's notion of Σ-persistent completeness seems (at least superficially) somewhat more akin to the ideas in this section than to those in Section 2.

REFERENCES

[1] A. Robinson, *Infinite forcing in model theory,* Proceedings of the Second Scandinavian Logic Symposium, North Holland, Amsterdam - London 1971.

[2] D. Saracino, *Generic structures and model companions,* Doctoral Dissertation, Princeton 1972.

[3] — *Infinite forcing in model theory and m-existentially closed structures,* Notices of the American Mathematical Society, June 1971, p. 668.

PRINCETON UNIVERSITY

Reçu par la Rédaction le 25. 11. 1972