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VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY SOLVABLE AND NILPOTENT GROUPS

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1. This paper is a sequel to [6]. In [2] it was shown that any connected Lie group in a variety of topological groups generated by solvable (nilpotent) connected Lie groups is solvable (nilpotent). This result was extended in [6] from solvable connected Lie groups to solvable connected locally compact groups. We prove a similar result here for the nilpotent case and also attack the connectedness condition.

Our notation and terminology will be that introduced in [6], and we will use the following basic result [1] on generating varieties:

THEOREM. *If Ω is any class of topological groups and G is a Hausdorff group in* $\mathbf{V}(\Omega)$, *then* $G \in \mathbf{SC}\overline{\mathbf{Q}}\overline{\mathbf{S}}\mathbf{D}(\Omega)$.

2. Our first theorem generalizes Theorem 2 of [6].

THEOREM 1. *If Ω is a class of locally compact groups each of which has the component of the identity solvable, then any connected locally compact group G in* $\mathbf{V}(\Omega)$ *is solvable.*

Proof. According to the theorem in Section 1, $G \in \mathbf{SC}\overline{\mathbf{Q}}\overline{\mathbf{S}}\mathbf{D}(\Omega)$; that is, G is a subgroup of a product $\prod_{i \in I} A_i$, where each $A_i \in \overline{\mathbf{Q}}\overline{\mathbf{S}}\mathbf{D}(\Omega)$.

We claim that each A_i has the property that the component of the identity is solvable. Firstly, we observe that each member of $\overline{\mathbf{S}}\mathbf{D}(\Omega)$ has this property. Let $B_i \in \overline{\mathbf{S}}\mathbf{D}(\Omega)$ and let f be an open continuous homomorphism of B_i onto A_i. If K_i and K_i' are the components of the identity in A_i and B_i, respectively, then, by Theorem 7.12 of [3], $f(K_i')$ is dense in K_i. Since K_i' is solvable, $f(K_i')$ is solvable and, consequently, K_i is solvable; that is, our claim has been proved.

Let $p_i(G)$ be the projection of G in A_i, and let G_i be the closure in A_i of $p_i(G)$. Since G is connected, $G_i \subset K_i$. Thus, each G_i is a solvable connected locally compact group. Finally, noting that $G \in \mathbf{SC}\{G_i : i \in I\}$, we infer, by Theorem 2 of [6], that G is solvable.

COROLLARY. *If Ω is a class of solvable locally compact groups and G is a connected locally compact group in $\mathbf{V}(\Omega)$, then G is solvable.*

Remark. We note that if the connectedness restriction on G is removed, then the resulting proposition is false. For example, let A_m, $m = 1, 2, \ldots$, be discrete finite solvable groups such that each A_m is of solvability length not less than m. Then the product group $\prod_{m=1}^{\infty} A_m$ is a compact non-solvable group in $\mathbf{V}\{A_m : m = 1, 2, \ldots\}$. However, we shall see that we can drop the connectedness restriction on G if we insist instead that G be a Lie group.

LEMMA 2. *If Ω is any class of topological groups and G is a discrete in $\mathbf{V}(\Omega)$, then $G \in \overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}(\Omega)$.*

Proof. We will show that $G \in \overline{\mathbf{S}}\mathbf{D}\overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}(\Omega)$. This suffices as clearly [1]

$$\overline{\mathbf{S}}\mathbf{D}\overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}(\Omega) \subset \overline{\mathbf{S}}\,\overline{\mathbf{Q}}\mathbf{D}\overline{\mathbf{S}}\mathbf{D}(\Omega) \subset \overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}\overline{\mathbf{S}}\mathbf{D}(\Omega) = \overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}(\Omega).$$

If we put $\Gamma = \overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}(\Omega)$, then, using the theorem in Section 1, we infer that $G \in \mathbf{SC}(\Gamma)$; that is, G is a subgroup of a product $\prod_{i \in I} F_i$, where each F_i is in Γ. Since G is discrete, there exist $a_1, \ldots, a_n$ in I such that

$$e = \Big(\prod_{i \in I} O_i\Big) \cap G,$$

where e is the identity element of G, each O_i is an open set in F_i and, for $i \neq a_j$ and some $j \in \{1, \ldots, n\}$, $O_i = F_i$. Let $F = \prod_{j=1}^{n} F_{a_j}$ and let p be the natural projection mapping of $\prod_{i \in I} F_i$ onto F. It is readily seen that $p(G)$ is isomorphic to G. Thus $G \in \mathbf{SD}(\Gamma)$. Finally, we note that since G is complete, $G \in \overline{\mathbf{S}}\mathbf{D}(\Gamma)$.

THEOREM 3. *If Ω is a class of solvable locally compact groups, then any Lie group G in $\mathbf{V}(\Omega)$ is solvable.*

Proof. Let C be the component of the identity in G. Since C is a connected Lie group in $\mathbf{V}(\Omega)$, by Theorem 1 we infer that C is solvable. Now, G/C is a totally disconnected Lie group, and hence it is discrete. Therefore, by Lemma 2, $G/C \in \overline{\mathbf{Q}}\,\overline{\mathbf{S}}\mathbf{D}(\Omega)$ and, consequently, must be solvable. Finally, since C and G/C are solvable, so G is too.

THEOREM 4. *If Ω is a class of locally compact groups each of which has the component of the identity nilpotent, then any connected locally compact group in $\mathbf{V}(\Omega)$ is nilpotent.*

Proof. According to Section 4.6 of [5], G has a compact normal subgroup N such that G/N is a Lie group. Further, by Section 4.13 of [5], N is contained in a compact connected subgroup M of G. Since

$M \in \mathbf{V}(\Omega)$, by Theorem 1 of [6] we infer that M is abelian. Thus N is a compact normal abelian subgroup of G, and, consequently, [4] it is central. We now prove that G/N is nilpotent, and hence G is nilpotent.

Now suppose that each member of Ω is connected. Then, as in Theorem 2 of [6], $\mathbf{V}(\Omega) = \mathbf{V}(\Gamma)$ for some class Γ of connected nilpotent Lie groups. Since G/N is a connected Lie group in $\mathbf{V}(\Gamma)$, by Corollary 3.4 of [2] we infer that G/N is nilpotent. Finally, note that the connected restriction on Ω can be removed as in Theorem 1.

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