

MARCZEWSKI INDEPENDENCE IN MONO-UNARY ALGEBRAS

BY

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By a *mono-unary algebra* we mean a pair (A, f) , where A is a non-empty set and f is a function from A into A . For (A, f) a mono-unary algebra and X a subset of A , we denote by $\text{Sg} X$ the subuniverse of (A, f) generated by X , i.e. the smallest subset of A which contains X and is closed under f . By $|X|$ we mean the cardinal number of X ; the empty set is denoted by 0 . For f a function and n a non-negative integer we define f^n recursively by $f^0 x = x$ and $f^{n+1} x = ff^n x$. For R a relation on a set X and $x \in X$, we define x/R to be the set of all y such that $(x, y) \in R$.

Given an algebra (A, f) , a set $X \subseteq A$ is called *independent* if and only if given any function g from X into A there is a homomorphism h from $(\text{Sg} X, f)$ into (A, f) such that $hx = gx$ for all $x \in X$. This notion is due to Marczewski; for a survey of the study of this notion of independence, the reader is referred to [2], where additional references may be found. The main result of this paper ⁽¹⁾ is Theorem 2.1 in which we characterize the family of independent sets of a unary algebra.

In section 2 of [1], Marczewski gives a necessary and sufficient condition for a set X to be independent. In the case of a mono-unary algebra (A, f) , this reduces to the following:

(MC) For $X \subseteq A$ to be independent it is necessary and sufficient that

(i) for any $x \in X$ and natural numbers m and n , if $f^m x = f^n x$, then $f^m a = f^n a$ for all $a \in A$, and

(ii) for any two distinct elements x and y of X and any two natural numbers m and n , if $f^m x = f^n y$, then $f^m a = f^n b$ for all a and b in A .

From (MC) it follows that if **Ind** is the family of independent sets of a mono-unary algebra, then **Ind** satisfies

(F) $X \in \mathbf{Ind}$ if and only if for every $Y \subseteq X$ with $|Y| \leq 2$, $Y \in \mathbf{Ind}$.

⁽¹⁾ Research supported by National Aeronautics and Space Administration Traineeship and National Science Foundation Traineeship.

The author is indebted to Professor Donald Monk for making available some unpublished notes on this subject as well as for reading the first draft of this paper.

1. Independent sets in mono-unary algebras. Throughout this section we assume that (A, f) is a fixed but arbitrary mono-unary algebra and that **Ind** is the family of independent sets in A . Before proceeding, we need some special concepts.

Definition 1.1. For $x \in A$

(i) lx (loop degree of x) is the smallest positive integer n for which there is a non-negative integer m with $f^m x = f^{m+n} x$. If no such n exists, then $lx = \infty$.

(ii) hx (height of x) is the smallest non-negative integer m for which there is a positive integer n with $f^m x = f^{m+n} x$. If no such m exists, $hx = \infty$.

(iii) x is a loop element if and only if $hx = 0$.

(iv) lA is the least common multiple of $\{lx : x \in A\}$ if such a number exists, otherwise, $lA = \infty$.

(v) hA is the (possibly infinite) least upper bound of $\{hx : x \in A\}$.

Some basic properties of these concepts are given in

LEMMA 1.2. For all $x \in A$

(i) $lx = lfx$.

(ii) $hfx = \max\{0, hx - 1\}$. (Here $\infty - 1 = \infty$.)

(iii) $lx = \infty$ if and only if $hx = \infty$.

(iv) $f^n x$ is a loop element if and only if $n \geq hx$.

Definition 1.3. (i) A subalgebra (B, f) of (A, f) is *connected* if and only if for every $x, y \in B$, there are natural numbers m and n with $f^m x = f^n y$.

(ii) A maximal connected subalgebra of (A, f) is called a *component*.

LEMMA 1.4. For $x \in A$, $\{x\} \in \mathbf{Ind}$ if and only if $hx = hA$ and $lx = lA$.

Proof. It immediately follows from (MC) that if $hx = hA$ and $lx = lA$ then $\{x\} \in \mathbf{Ind}$. Now if $hx < hA$, there is a $y \in A$ with $hy > hx$. But then $f^{hx} x = f^{hx+lx} x$, while $f^{hx} y \neq f^{hx+lx} y$, whence, by (MC), $\{x\} \notin \mathbf{Ind}$. A similar argument shows that if $lx < lA$, then $\{x\} \notin \mathbf{Ind}$.

LEMMA 1.5. If x and y are distinct elements of A which belong to the same component of (A, f) and $\{x, y\} \in \mathbf{Ind}$, then (A, f) is connected, $lA = 1$, $f^m x = f^n y$ implies $m, n \geq hA$, and hA is finite.

Proof. Since x, y belong to the same component of A , there are natural numbers m and n with $f^m x = f^n y$. Then by (MC), $f^m a = f^n b$ for all $a, b \in A$, thus (A, f) is connected. Furthermore, $f^m a = f^n b = f^m f a = f^{m+1} a$, so $lA = 1$. Now whenever $f^m x = f^n y$, we have $f^m x = f^n f y$, so $n \geq hy = hA$. The last statement follows from the others.

THEOREM 1.6. *Suppose x and y are distinct elements of A . Then $\{x, y\} \in \mathbf{Ind}$ if and only if $hx = hy = hA$, $lx = ly = lA$, and either (i) x and y belong to different components of (A, f) , or (ii) A is connected, $lA = 1$, and $f^m x = f^n y$ implies $m, n \geq hA$.*

Proof. The necessity of the conditions follows from 1.4 and 1.5. Conversely, suppose $hx = hy = hA$ and $lx = ly = lA$. If (i) holds and $f^m w = f^n z$, where $\{w, z\} \subseteq \{x, y\}$, then $w = z$, thus it follows from 1.4 that (MC) holds for $\{x, y\}$. If (ii) holds, then $\{x, y\}$ clearly satisfies (MC).

Condition (F) together with 1.6 already gives us a fairly clear picture of the appearance of **Ind**. There is one more important condition on **Ind** which we must establish. Let U be the set of all x such that $\{x\} \in \mathbf{Ind}$, and let R be the set of all pairs (x, y) with $x, y \in U$ and either $x = y$ or $\{x, y\} \notin \mathbf{Ind}$.

THEOREM 1.7. *R is an equivalence relation with field U .*

Proof. Obviously R is reflexive and symmetric and has field U . To show R is transitive, suppose $\{x, y\} \notin \mathbf{Ind}$ and $\{x, z\} \in \mathbf{Ind}$; we will show that $\{y, z\} \in \mathbf{Ind}$. If (A, f) is not connected, x and z must belong to different components, while x and y belong to the same component. Thus y and z must belong to different components and so, by 1.6, $\{y, z\} \in \mathbf{Ind}$. Now suppose that (A, f) is connected; then, by 1.6, there are natural numbers m and n with $f^m x = f^n y$ and at least one of m and n is less than hA , say $m < hA$. Then $hf^n y = hf^m x = hx - m \neq 0$. Thus $hy - n = hx - m$; but $hx = hy$, so $m = n$. Now suppose that $\{y, z\} \notin \mathbf{Ind}$; then by an argument similar to the one above, there is an $s < hA$ with $f^s y = f^s z$. Letting $t = \max\{n, s\}$, we have $f^t x = f^t z$ and $t < hA$. This contradicts the assumption that $\{x, z\} \in \mathbf{Ind}$, thus $\{y, z\} \in \mathbf{Ind}$.

2. Characterization of the family of independent sets. If $0 < |A| \leq 3$ and **Ind** is a family of subsets of A , it is easy to see that **Ind** is the family of independent sets of a mono-unary algebra (A, f) if and only if **Ind** contains a non-empty set and satisfies condition (F). Throughout the remainder of this section we assume that A has at least four elements, **Ind** is a family of subsets of A , and U is the set of all x such that $\{x\} \in \mathbf{Ind}$. Further, we assume that R is the set of all pairs (x, y) with x and y elements of U and either $x = y$ or $\{x, y\} \notin \mathbf{Ind}$. We are now ready to state the main theorem of this paper.

THEOREM 2.1. ***Ind** is the family of independent sets of a mono-unary algebra (A, f) if and only if **Ind** satisfies condition (F) and R is an equivalence satisfying at least one of the following three conditions:*

- (i) *For each $x \in U$, x/R is infinite.*
- (ii) *For all $x, y \in U$, $|x/R| = |y/R|$; if $|x/R| = 1$, then $|A \sim U| \leq 1$.*
- (iii) *$|\{x/R : x \in U\}| \leq |A \sim U|$.*

Note that it is not claimed that (i), (ii), and (iii) are mutually exclusive; in fact, it is possible that all three hold simultaneously. The remainder of this section is devoted to establishing 2.1. First recall from section 1 that if **Ind** is the family of independent sets in a mono-unary algebra (A, f) , then **Ind** satisfies condition (F) and R is an equivalence relation. The next three lemmata show that at least one of the three numbered conditions is also satisfied.

LEMMA 2.2. *Assume **Ind** is the family of independent sets in a mono-unary algebra (A, f) and $hA = \infty$. Then for each $x \in U$, x/R is infinite.*

Proof. Let $x \in U$. Then by 1.4, $hx = \infty$, and so for each positive n , $hf^n x = \infty$; thus $f^n x \in U$. Now for n positive, $x \neq f^n x$, so by 1.6, $\{x, f^n x\} \notin \mathbf{Ind}$. Therefore $f^n x \in x/R$. Finally, for $m \neq n$, $f^m x \neq f^n x$, so x/R is infinite.

LEMMA 2.3. *Suppose **Ind** is the family of independent sets in a mono-unary algebra (A, f) and hA is positive but finite. Then $|\{x/R : x \in U\}| \leq |A \sim U|$ or $|A \sim U| = 1$ and $|x/R| = 1$ for all $x \in U$.*

Proof. Let X be a set containing exactly one element from each R class. By condition (F), $X \in \mathbf{Ind}$. Since $hA \neq 0$, we have $hfx < hx$ for $x \in X$, thus $fx \notin U$. Now if x and y are distinct elements of X and $fx = fy$, then by (MC), $fz = fw$ for all z and w in A ; it follows that (A, f) is connected and $hA = 1$. Applying 1.6, we obtain $|A \sim U| = 1$ and $|x/R| = 1$ for each $x \in U$. The only other possibility is that for all $x, y \in X$ with $x \neq y$, $fx \neq fy$. In this case $|\{fx : x \in U\}| = |X| = |\{x/R; x \in U\}|$. But $\{fx : x \in U\} \subseteq A \sim U$; the lemma follows.

LEMMA 2.4. *Suppose **Ind** is the family of independent sets in a mono-unary algebra (A, f) and $hA = 0$. Then for $x \in U$, $|x/R| = lA$; furthermore, if $lA = 1$, $U = A$.*

Proof. If $lA = 1$, then f is the identity on A , so the last statement holds. Now suppose $lA > 1$. For $x \in U$, $lfx = lx$ and $hfx = 0 = hA$, so by 1.4, $fx \in U$. Furthermore, by 1.6, $(x, f^n x) \in R$ for each natural number n . Now the sequence $x, fx, f^2 x, \dots$ contains exactly lA distinct elements, thus $|x/R| \geq lA$. On the other hand, if $(x, y) \in R$, it follows from 1.6 that there is an integer n with $f^n x = y$, thus $|x/R| \leq lA$.

We have shown that if **Ind** is the family of independent sets in (A, f) , then at least one of (i), (ii), (iii) of 2.1 holds. We now proceed in the other direction.

LEMMA 2.5. *Suppose **Ind** satisfies condition (F), R is an equivalence relation, and at least one of (i), (ii), (iii) of 2.1 is satisfied. Then there is a mono-unary algebra (A, f) having **Ind** as its family of independent sets.*

Proof. For each of the various possibilities we construct an algebra having **Ind** as its family of independent sets. The proofs that these examples do indeed have the proper family of independent sets are similar, so

by way of illustration, we shall do only the first one and leave the others to the reader. We will assume $U \neq 0$, the case where $U = 0$ being trivial.

Suppose condition (i) holds. For each R class, x/R , let t_x be a one-to-one function from the natural numbers into x/R . (If $(x, y) \in R$, then $t_x = t_y$.) Define f as follows; for $x \in U$, if there is n with $t_x n = x$, let $fx = t_x(n+1)$, otherwise let $fx = t_x 0$. For $x \notin U$, let $fx = x$. We claim that (A, f) has **Ind** as its family of independent sets. First observe that for $x \in A$, $x \in U$ if and only if $hx = \infty$, and by 1.4, this is equivalent to $\{x\}$ being independent. Now if x and y are distinct elements of U , $\{x, y\}$ is independent if and only if x and y belong to different components; this is equivalent to $(x, y) \notin R$, which in turn is equivalent to $\{x, y\} \in \mathbf{Ind}$. Now by condition (F) for both **Ind** and the family of independent sets, **Ind** is the family of independent sets in (A, f) .

Next suppose that (ii) holds and that for each $x \in U$, $n = |x/R|$ is greater than one but finite. For each R class x/R , let t_x be a one-to-one function from $\{0, 1, \dots, n-1\}$ into x/R . Define f as follows: For $x \in U$, there is m with $t_x m = x$; for $m < n-1$, let $fx = t_x(m+1)$; for $m = n-1$, let $fx = t_x 0$. For $x \notin U$, let $fx = x$. Then (A, f) is the required algebra.

If (ii) holds and x/R has a single element for each $x \in U$, we have two cases to consider. First, suppose $|A \sim U| = 0$; then let f be the identity on A . Second, suppose $A \sim U = \{z\}$; then let $fx = z$ for all $x \in A$. In either case, **Ind** is the family of independent sets in (A, f) .

Finally, suppose (iii) holds. We may assume $|A \sim U| > 1$, for if $|A \sim U| = 1$, then either (i) or (ii) also holds. Thus let t be a one-to-one function from $\{x/R : x \in U\}$ into $A \sim U$. For $x \in U$, let $fx = t(x/R)$; for $x \notin U$, let $fx = x$. Again (A, f) has **Ind** as its family of independent sets.

This completes the proof of 2.1.

REFERENCES

- [1] E. Marczewski, *Independence and homomorphisms in abstract algebras*, *Fundamenta Mathematicae* 50 (1961), p. 45-61.
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Reçu par la Rédaction le 6. 1. 1968