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ON $\mathcal{M}$-CONTRACTIBILITY

BY

TAKEMI MIZOKAMI (SASEBO)

1. Introduction. A space X is called *contractible* onto a metric space T or, briefly, *$\mathcal{M}$-contractible* if there exists a one-to-one continuous map from X onto T. $\mathcal{M}$-contractibility, its properties and characterizations have been studied by Martin [3].

In this paper, the author considers preserving the $\mathcal{M}$-contractibility by perfect maps and some relations to hyperspaces. In the sequel all spaces are assumed to be T_1, and N always denotes the set of positive integers.

2. Theorems. If $\mathcal{U}$ is an open cover of X, then $S(p, \mathcal{U}) = S^1(p, \mathcal{U})$ is the star of the point $p \in X$ in $\mathcal{U}$, and

$$S^k(p, \mathcal{U}) = S\big(S^{k-1}(p, \mathcal{U}), \mathcal{U}\big) \qquad \text{for } k > 1.$$

THEOREM 1. *For a space X the following are equivalent:*

(i) *X is $\mathcal{M}$-contractible.*

(ii) *There exists a sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X satisfying the following:*

(a)$_1$ $\{p\} = \bigcap_{n=1}^{\infty} S(p, \mathcal{U}_n)$ *for each* $p \in X$.

(b) *If* $U_1 \cap U_2 \neq \emptyset$ *for* $U_1, U_2 \in \mathcal{U}_{n+1}$, *then* $U_1 \cup U_2 \subset U_3$ *for some* $U_3 \in \mathcal{U}_n$.

(iii) *There exists a sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X satisfying* (b) *and, for each fixed $k \in N$,*

(a)$_k$ $\{p\} = \bigcap S^k(p, \mathcal{U}_n)$ *for each* $p \in X$.

Proof. (i) and (ii) are equivalent by Theorem 4.3 in [3].

(ii) $\Rightarrow$ (iii) Assume (a)$_{k-1}$, that is there exists a sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X satisfying (a)$_{k-1}$ and (b). Take $p \neq q \in X$. By (a)$_{k-1}$ there exists an $n \in N$ with $q \notin S^{k-1}(p, \mathcal{U}_n)$. Assume $q \in S^k(p, \mathcal{U}_{n+1})$. Then there exists a $U \in \mathcal{U}_{n+1}$ such that $q \in U$ and $U \cap S^{k-1}(p, \mathcal{U}_{n+1}) \neq \emptyset$. By (b), $U \subset S^{k-1}(p, \mathcal{U}_n)$. Thus $q \in S^{k-1}(p, \mathcal{U}_n)$, a contradiction.

Since (iii) $\Rightarrow$ (ii) is trivial, the proof is completed.

Frequently, a space with $\{\mathscr{U}_n\}$ satisfying $(\mathrm{a})_m$ is said to have a $G_\delta(m)$-*diagonal.* Of course, $G_\delta(1)$-diagonal is equivalent to the diagonal Δ of $X\times X$ being a G_δ. In the next part, $\mathscr{C}(X)$ denotes the hyperspace of all non-empty compact sets of X with the finite topology in the sense of [4]. Recall that $\mathscr{C}(X)$ is metrizable if X is metrizable.

THEOREM 2. *If X has a $G_\delta(m)$-diagonal, then $\mathscr{C}(X)$ has a $G_\delta(m-1)$-diagonal, where $m\geqslant 2$.*

Proof. Let $\{\mathscr{U}_n\}$ be a sequence of open covers such that $(\mathrm{a})_m$ is satisfied and $\mathscr{U}_{n+1}<\mathscr{U}_n$ for each n. For each n construct

$$\langle\mathscr{U}_n\rangle=\{\langle U_1,\dots,U_k\rangle:\ U_1,\dots,U_k\in\mathscr{U}_n,\ k\in N\}.$$

Then each $\langle\mathscr{U}_n\rangle$ is an open cover of $\mathscr{C}(X)$. Let K_1 and K_2 be distinct points in $\mathscr{C}(X)$. Without loss of generality assume $p\in K_1-K_2$. There exists an $n\in N$ with $p\notin S^{m-1}(K_2,\mathscr{U}_n)$. Indeed, if $x_n\in S^{m-1}(p,\mathscr{U}_n)\cap K_2$ for each n, then $\{x_n\}$ has a cluster point in $K_2\cap\bigcap_{n=1}^{\infty}\overline{S^{m-1}(p,\mathscr{U}_n)}$. Since $\{\mathscr{U}_n\}$ satisfies $(\mathrm{a})_m$,

$$\{p\}=\bigcap_{n=1}\overline{S^{m-1}(p,\mathscr{U}_n)}.$$

This is a contradiction. It is routine to see that $K_1\notin S^{m-1}(K_2,\langle U_n\rangle)$. Thus $\mathscr{C}(X)$ has a $G_\delta(m-1)$-diagonal.

THEOREM 3. *A space X is $\mathscr{M}$-contractible if and only if $\mathscr{C}(X)$ is $\mathscr{M}$-contractible.*

Proof. Since X is homeomorphic to $\{\{x\}:x\in X\}\subset\mathscr{C}(X)$, the "if" part is trivial. To see the "only if" part, let $\{\mathscr{U}_n\}$ be a sequence of open covers satisfying $(\mathrm{a})_2$ and (b). Construct $\{\langle\mathscr{U}_n\rangle:\ n\in N\}$ in the same fashion as in the proof of Theorem 2. Then $\{\langle\mathscr{U}_n\rangle\}$ satisfies $(\mathrm{a})_1$ by Theorem 1.

It remains to prove that (b) is satisfied. Assume

$$(*)\qquad K\in\langle U_1,\dots,U_s\rangle\cap\langle V_1,\dots,V_t\rangle\neq\emptyset,$$

where $U_1,\dots,U_s,V_1,\dots,V_t\in\mathscr{U}_{n+1}$. Let

$$A=\{(i,j):\ U_i\cap U_j\neq\emptyset\}.$$

Then for each $(i,j)\in A$ there exists a $W_{(i,j)}\in\mathscr{U}_n$ with $U_i\cup V_j\subset W_{(i,j)}$. Construct

$$\mathscr{W}=\langle W_{(i,j)}\rangle_{(i,j)\in A}\in\langle\mathscr{U}_n\rangle.$$

Suppose $L\in\langle U_1,\dots,U_s\rangle$. Then each point $x\in L$ belongs to some U_i which, by $(*)$, intersects some V_j. Thus $x\in W_{(i,j)}$, implying

$$L\subset\bigcup\{W_{(i,j)}:(i,j)\in A\}.$$

It is obvious that $L \cap W_{(i,j)} \neq \emptyset$ for each $(i, j) \in A$. This is the case also if $L \in \langle V_1, \dots, V_t\rangle$. Hence we have

$$\langle U_1, \dots, U_s\rangle \cup \langle V_1, \dots, V_t\rangle \subset \mathscr{W}.$$

This completes the proof.

Let $\mathscr{C}[X]$ be the set of all non-empty compact sets of X with the Pixley-Roy topology in [2] which is finer than the topology of $\mathscr{C}(X)$ by Proposition 2.1 in [2]. From Theorem 2 it follows that if X is $\mathcal{M}$-contractible, then so is $\mathscr{C}[X]$.

THEOREM 4. *Let f be an open perfect map from X onto Y. If X is $\mathcal{M}$-contractible, then so is Y.*

Proof. Define $g: \mathscr{C}(Y) \to \mathscr{C}(X)$ by

$$g(K) = f^{-1}(K) \quad \text{for } K \in \mathscr{C}(Y).$$

CLAIM. *g is continuous.*

Suppose $g(K) = f^{-1}(K) \in \langle V_1, \dots, V_k\rangle = \mathscr{V}$. Let

$$W = Y - f(X - V_1 \cup \dots \cup V_k), \qquad \mathscr{W} = \langle W \cap f(V_1), \dots, W \cap f(V_k)\rangle.$$

Then $\mathscr{W}$ is an open neighborhood of K such that $g(\mathscr{W}) \subset \mathscr{V}$. Thus g is continuous.

Since g is one-to-one and onto, there exists a contraction from $\mathscr{C}(Y)$ onto an $\mathcal{M}$-contractible (by Theorem 2) $\mathscr{C}(X)$. Consequently, $\mathscr{C}(Y)$ is $\mathcal{M}$-contractible and, therefore, by Theorem 2, so is Y.

This gives an affirmative answer to the question posed by Martin in [3]: do perfect open maps preserve $\mathcal{M}$-contractibility?

Martin asked in [3] also the following question:

If f is a perfect map from an $\mathcal{M}$-contractible space X onto Y such that

(**) the family of all non-trivial fibers is discrete in X, then must Y be $\mathcal{M}$-contractible?

Burke's example in [1] shows that even if f is a perfect map with property (**) from a space with a G_δ-diagonal onto Y, then Y need not have a G_δ-diagonal. On the other hand, Popov's example in [5] shows that even if f is a perfect map from an $\mathcal{M}$-contractible space X onto Y, then Y need not have a G_δ-diagonal. However, concerning the preservation of G_δ-diagonals by perfect maps, we can establish the following results by combining these properties:

THEOREM 5. *Let f be a perfect map from X onto Y such that (**) is satisfied. If X has a $G_\delta(2)$-diagonal and the union of all non-trivial fibers is G_δ in X, then Y has a G_δ-diagonal.*

Proof. Since X has a $G_\delta(2)$-diagonal, each compact set $f^{-1}(y)$ is a G_δ-set. By assumption,

$$\bigcup\{f^{-1}(y):\ y \in Y_2\} = \bigcap_{n=1}^{\infty} W_n,$$

where W_n is an open set for each $n \in N$ such that $W_{n+1} \subset W_n$ for each $n \in N$ and

$$Y_2 = \{y \in Y:\ f^{-1}(y) \text{ is non-trivial, i.e., } |f^{-1}(y)| \geqslant 2\}.$$

By $(**)$ there exists an open set $P(y)$ such that

$$P(y) \cap \bigcup\{f^{-1}(y'):\ y' \in Y_2, y \neq y'\} \neq \emptyset, \quad f^{-1}(y) \subset P(y).$$

Write

$$f^{-1}(y) = \bigcap_{n=1}^{\infty} V_n(y),$$

where $V_n(y)$ is an open set such that

$$V_{n+1}(y) \subset V_n(y) \subset P(y) \cap W_n \quad \text{for each } n \in N.$$

Put

$$V'_n(y) = Y - f\big(X - V_n(y)\big) \quad \text{and} \quad W'_n = Y - f(X - W_n).$$

Now let $\{\mathscr{U}_n : n \in N\}$ be a sequence of open covers such that $(a)_1$ is satisfied and $\mathscr{U}_{n+1} < \mathscr{U}_n$ for each $n \in N$. Put

$$\mathscr{V}_n = \{Y_1 \cap \big(Y - f(X - U)\big) : U \in \mathscr{U}_n\} \quad \text{and} \quad \mathscr{W}_n = \mathscr{V}_n \cup \{V'_n(y) : y \in Y_2\},$$

where $Y_1 = Y - Y_2$. Then $\mathscr{W}_n$ is an open cover of Y. Let p and q be distinct points in Y. If $p, q \in Y_1$, then there exist $m, n \in N$ with $p, q \notin W'_m$ and $p \notin S(q, \mathscr{V}_n)$. Then we have $p \notin S(q, \mathscr{W}_k)$ for $k = \max(m, n)$. If $p, q \in Y_2$, then $p \notin S(q, \mathscr{W}_n) = V'_n(q)$ for every n. If $p \in Y_1$ and $q \in Y_2$, then there exists an $m \in N$ such that $p \notin W'_m$. Thus we obtain $p \notin S(q, \mathscr{W}_m) = V'_m(q)$. Hence Y has a G_δ-diagonal.

COROLLARY 1. *Let $f : X \to Y$ be a perfect map with property $(**)$. If X is a perfect space (i.e., every closed set is a G_δ-set) with a G_δ-diagonal, then Y has a G_δ-diagonal.*

This can be proved by the repetition of the essential part of the proof of Theorem 5.

COROLLARY 2. *Let f be the same as in Corollary 1. If X is $\mathscr{M}$-contractible and the union of all non-trivial fibers is G_δ in X, then Y has a G_δ-diagonal.*

COROLLARY 3. *Let $f : X \to Y$ be the same as in Corollary 1. If X is a paracompact perfectly normal and $\mathscr{M}$-contractible space, then so is Y.*

THEOREM 6. *Let $f : X \to Y$ be an open compact map with property $(**)$. If X is $\mathscr{M}$-contractible, then so is Y.*

Proof. Let φ be a contraction from X onto a metric space T. Define $g\colon Y \to \mathscr{C}(T)$ by $g(y) = \varphi(f^{-1}(y))$ for each $y \in Y$. Since φ is one-to-one, so is g. We shall show that g is continuous. Suppose

$$g(y) = \varphi(f^{-1}(y)) \in \mathscr{V} = \langle V_1, \ldots, V_k\rangle \cap g(Y),$$

where $V_1, \ldots, V_k$ are open sets in T. Then $f^{-1}(y) \in \langle\varphi^{-1}(V_1), \ldots, \varphi^{-1}(V_k)\rangle$. Let Y_1 and Y_2 be the same as in the proof of Theorem 5.

Case 1. If $y \in Y_1$, then $f^{-1}(y)$ is a single point x such that

$$x \in \bigcap_{j=1}^{k} \varphi^{-1}(V_j).$$

Then

$$y \in f\Big(\bigcap_{j=1}^{k} \varphi^{-1}(V_j)\Big) \cap Y_1 = O.$$

It is obvious that O is an open set such that $g(O) \subset \mathscr{V}$.

Case 2. If $y \in Y_2$, then there exists an open set $P(y)$ such that

$$f^{-1}(y) \subset P(y) \subset \bigcup_{j=1}^{k} \varphi^{-1}(V_j), \qquad P(y) \cap \bigcup \{f^{-1}(y') : y' \neq y,\, y' \in Y_2\} = \emptyset.$$

Set

$$O = f(P(y)) \cap \bigcap_{j=1}^{k} f(\varphi^{-1}(V_j)).$$

Then O is an open set containing y such that $g(O) \subset \mathscr{V}$. This completes the proof.

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