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LINE MINIMAL BOOLEAN FORESTS

BY

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The relationship between groups and graphs has been studied intensively (see [3], Chapter 14). Frucht [1] showed that, for any finite group A, there are infinitely many non-isomorphic cubic graphs whose automorphism groups are isomorphic to A. This result was later generalized by Sabidussi [8] to include other classes of graphs. Some attempts have been made to characterize the groups of special classes of graphs. Kagno [4] catalogued the groups of all graphs with at most six points; both Pólya [6] and Prins [7] characterized the automorphism groups of trees. Our object* is to study a class of graphs which have a specified group structure, namely, those graphs whose non-trivial automorphisms are all of order 2.

For graph theoretic terms not defined here, see [3]. We shall denote the (automorphism) group of a graph G by $\Gamma(G)$. A group Γ is *boolean* if every non-trivial element has order 2. This terminology is classical and is mentioned in an early paper by one of us [2]. A *boolean graph* has a boolean group; a graph whose group consists of the identity permutation alone is an *identity graph*. A *fixed point* of a graph G is invariant under every automorphism of G.

General properties of boolean graphs. All boolean graphs with $p \leqslant 5$ points are drawn in the Appendix. Following standard notation, let S_p be the symmetric group of degree p and let $A[B]$ denote the wreath product of two permutation groups A and B. As in [3], p. 163, let $A+B$ denote their direct sum.

THEOREM 1. *A graph G is boolean if and only if every component is boolean, and all the components are distinct except, possibly, for pairs of isomorphic identity graphs.*

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Proof. Let G be given by

$$G = k_1G_1 \cup k_2G_2 \cup \ldots \cup k_nG_n,$$

where the components G_i $(1 \leqslant i \leqslant n)$ are distinct and each $k_i \geqslant 1$. Then as in [3], p. 166,

$$\Gamma(G) = S_{k_1}[\Gamma(G_1)] + \ldots + S_{k_n}[\Gamma(G_n)].$$

Clearly, $\Gamma(G)$ is boolean if and only if each $S_{k_j}[\Gamma(G_j)]$ is boolean. Moreover, $S_{k_j}[\Gamma(G_k)]$ is boolean if and only if $k_j = 1$ and G_j is boolean or $k_j = 2$ and G_j is an identity graph.

COROLLARY 1a. *Let G and H be connected graphs. Then their union is boolean if and only if G and H are distinct boolean graphs or both are identity graphs.*

COROLLARY 1b. *Let G and H be boolean graphs. Then their join $G+H$ is boolean if and only if the components of $\bar{G} \cup \bar{H}$ are distinct except, possibly, for pairs of identity graphs.*

Proof. It suffices to note that $G+H = \overline{\bar{G} \cup \bar{H}}$ (see Zykov [9]), and hence $\Gamma(G+H) = \Gamma(\bar{G} \cup \bar{H})$.

It is of interest in connection with boolean graphs to relate the structure of the adjacency matrix to the automorphism group. The *adjacency matrix* $A = A(G) = (a_{ij})$ of a graph G with points $v_1, v_2, \ldots, v_p$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The next result is given in [5]. Recall that the eigenvalues of a real symmetric matrix are real.

THEOREM 2. *If the eigenvalues of $A(G)$ are distinct, then $\Gamma(G)$ is boolean.*

The converse of this theorem is not true, as demonstrated by the graph in Fig. 1 which is boolean but, nevertheless, has -1 as a double eigenvalue, as can be verified at once from its characteristic polynomial.

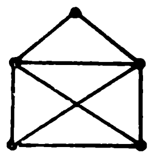


Fig. 1. A boolean graph without distinct eigenvalues

Boolean trees and forests. The following characterization of boolean trees is an immediate consequence of a theorem of Prins [7], p. 48.

THEOREM 3. *A tree T is boolean if and only if exactly one of the following conditions is satisfied:*

(i) *If T has no fixed point, then $\Gamma(T) \cong S_2$.*

(ii) *If T has a fixed point, then, at each point, T has at most two similar branches and both such branches in any pair are isomorphic rooted identity trees.*

Note that $\Gamma(T)$ is isomorphic to S_2 if and only if T consists of two isomorphic rooted identity trees with a line joining their roots.

A *minimal boolean graph* with p points has the smallest possible number of lines. According to Theorem 1, a forest F is boolean if and only if all the trees of F are distinct boolean trees except, possibly, for pairs of identity trees. The concept of a boolean forest is of interest in connection with minimal boolean graphs, although such a graph is not necessarily a forest; see G_3 in Fig 2.

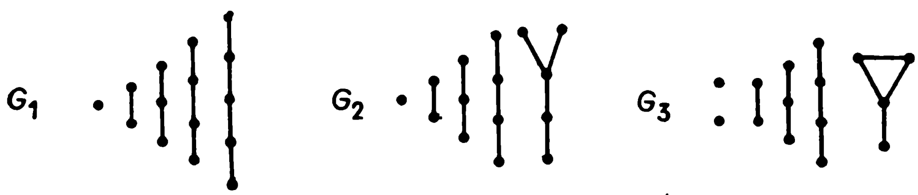


Fig. 2. Three minimal boolean graphs with 15 points

A *linear forest* (for example, G_1 of Fig. 2) is a forest whose components are paths. Let P_n be the path with n points. Since every path is boolean and (except for P_1) has a non-trivial automorphism, it follows from Theorem 1 that a linear forest is boolean if and only if it consists of paths of distinct lengths except, possibly, for two isolated points. We turn now to a characterization of minimal boolean linear forests (mblf).

Note first that to each boolean linear forest with p points there corresponds a unique partition of the integer p whose parts represent the number of points in the respective paths. Moreover, every partition of an integer p into distinct parts except, possibly, for two 1's corresponds to a unique boolean linear forest. Thus, we shall speak of boolean linear forests and boolean partitions interchangeably, and assume that the parts of a boolean partition are arranged in order of non-decreasing size. If $r_1 \leqslant r_2 < \ldots < r_k$ is the partition corresponding to a boolean linear forest F with p points, the number q of lines of F is

$$q = \sum_{i=1}^{k} (r_i - 1) = p - k.$$

By the *size* of a boolean partition we shall mean the number of lines in the corresponding boolean linear forest.

THEOREM 4. *Let $\pi(n)$ denote the number of partitions of the integer n, and let*

$$k = \left[\frac{1+\sqrt{8p-7}}{2}\right].$$

Then every p-point minimal boolean linear forest has the size $s = p-k$. Furthermore, the number of such mblf's is equal to

$$\pi\left(p-1-\binom{k}{2}\right).$$

Proof. We note that, by definition, k satisfies

$$\frac{-1+\sqrt{8p-7}}{2} < k \leqslant \frac{1+\sqrt{8p-7}}{2}. \tag{1}$$

Now, if there exists a boolean linear forest of size $s \leqslant p-k-1$, the points must be partitioned into $k+1$ paths, two of which may be isolates, but the remaining paths must each have a different length. Consequently, we must have

$$p \geqslant 1+\sum_{i=1}^{k} i = 1+\frac{k(k+1)}{2}.$$

Substituting the lower bound for k from (1) yields

$$p > 1+\frac{1}{2}\left(\frac{-1+\sqrt{8p-7}}{2}\right)\left(\frac{1+\sqrt{8p-7}}{2}\right) = p,$$

obviously a contradiction.

On the other hand, an mblf of size $s = p-k$ is given by selecting paths of order

$$r_1 = 1, \quad r_i = i-1 \text{ for } 2 \leqslant i \leqslant k-1, \quad \text{and} \quad r_k = p-\sum_{i=1}^{k+1} r_i.$$

This forest is boolean provided $r_k \geqslant k-1$. But, in fact, we observe that

$$r_k = p-1-\binom{k-1}{2} \geqslant p-1-\frac{1}{2}\left(\frac{-1+\sqrt{8p-7}}{2}\right)\left(\frac{-3+\sqrt{8p-7}}{2}\right)$$
$$= \frac{-1+\sqrt{8p-7}}{2} \geqslant k-1.$$

Thus, we have constructed an mblf of size $p-k$.

To count the number of mblf's, we shall establish a one-to-one correspondence between them and partitions of $p-1-\binom{k}{2}$. Every mblf has an isolate, for, if not, the removal of an endline from the smallest path would produce an mblf with one less line. Thus, if $r_1, r_2, \ldots, r_k$ is the partition of this mblf, then r_1 must be 1, and the remaining r_i's provide a partition of $p-1$ into $k-1$ distinct positive parts. Consequently, each $r_i < r_{i+1}$ for $2 \leqslant i \leqslant k-1$. Let $n_i = r_{i+1}-i$ for $1 \leqslant i \leqslant k-1$. Now we see that

$$\sum_{i=1}^{k-1} n_i = p-1-\binom{k}{2}.$$

Furthermore, we claim that these n_i's are arranged in non-decreasing order, for, if $n_i < n_{i-1}$, then

$$r_{i+1}-i < r_i - i + 1 \quad \text{or} \quad r_{i+1} \leqslant r_i$$

contradicting $r_{i+1} > r_i$. Thus, from each mblf we construct a unique partition of $p-1-\binom{k}{2}$.

Conversely, we observe that

$$p-1-\binom{k}{2} < p-1-\frac{1}{2}\left(\frac{-1+\sqrt{8p-7}}{2}\right)\left(\frac{-3+\sqrt{8p-7}}{2}\right)$$

$$= \frac{-1+\sqrt{8p-7}}{2} < k.$$

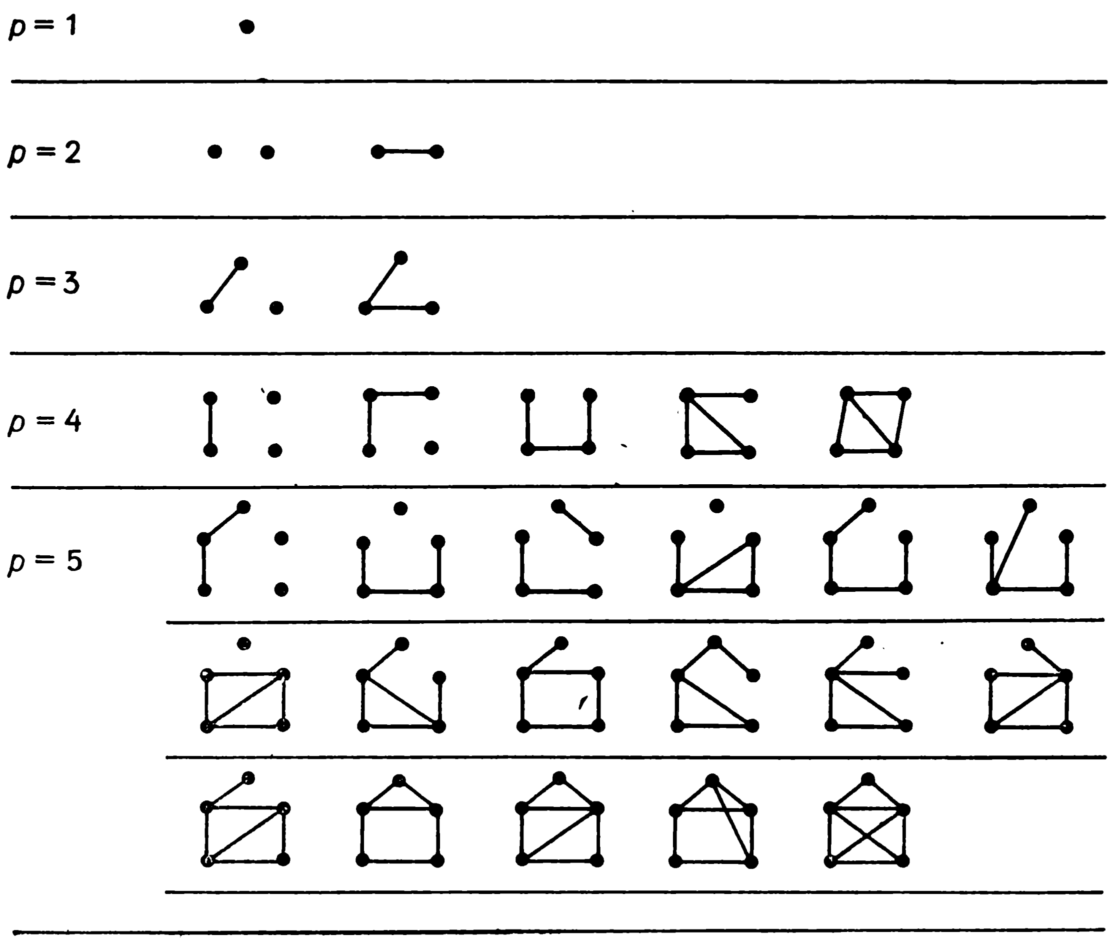
Thus, any partition of $p-1-\binom{k}{2}$ has at most $k-1$ parts, so we may reverse the above procedure in the following manner. Take any partition of $p-1-\binom{k}{2}$ arranged in non-decreasing order. If necessary, insert leading terms of zero until there are $k-1$ terms in the sequence $0 \leqslant n_1 \leqslant n_2 \leqslant \ldots \leqslant n_{k-1}$. Define $r_1 = 1$ and $r_{i+1} = i + n_i$ for $1 \leqslant i \leqslant k-1$.

Clearly, the r_i's form a boolean partition of p, completing the correspondence. Consequently, the number of minimal boolean forests is given by

$$\pi\left(p-1-\binom{k}{2}\right)$$

as asserted.

Appendix. Boolean graphs with at most 5 points.



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