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AN EXAMPLE OF A WEAK BERNOULLI PROCESS WHICH IS NOT FINITARY

BY

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Introduction. Suppose that a proces is known to be a stationary irreducible aperiodic Markov chain with a finite number of states, but for some reason the states cannot be directly observed: the states of the process are partitioned into groups and one can only identify the group from which an observation came. The observed process in then called a *functional of the Markov chain.*

The problem considered by Gilbert [3] was to characterize functionals of finite Markov chains. In [4] Heller did it in terms of certain finite dimensional modules and introduced stochastic processes called *finitary processes*, a class including functionals of Markov chains. Robertson has shown in [6] that every finitary and mixing process is a Kolmogoroff process. It follows from [1] and [2] that every finitary and totally ergodic process is in fact weak Bernoulli.

In this paper we give an example of a weak Bernoulli process which is not finitary. We do this by means of the Robertson representation of a stochastic process.

1. Preliminaries. Let $(X, \mathscr{B}, m)$ be a Lebesgue probability space, T be an automorphism of $(X, \mathscr{B}, m)$ with finite entropy, and $Q = \{Q_i\colon i \in I\}$ be a finite measurable partition of X.

Let us recall that the process (T, Q) is *weak Bernoulli* if for every $\varepsilon > 0$ there exists a positive integer N such that for all $n \geqslant N$, $k \geqslant 1$ the partitions $\bigvee_{0}^{k} T^{-i}Q$ and $\bigvee_{n+k}^{n+2k} T^{-i}Q$ are ε-independent, i.e.,

$$\sum |m(A \cap B) - m(A)m(B)|,$$

where the sum is taken over all $A \in \bigvee_{0}^{k} T^{-i}Q$ and $B \in \bigvee_{n+k}^{n+2k} T^{-i}Q$.

Now we recall the concept of a spectral representation of a stochastic process introduced by Robertson in [5].

Let H be a complex Hilbert space, e be a fixed element of H with $\|e\| = 1$, and $\{W_i: i \in I\}$ be a finite family of contractions on H. Let

$$W = \sum_{i \in I} W_i$$

and let W^* denote the linear operator adjoint with W.

Let

$$I_\infty = \bigcup_{n=0}^{\infty} I^n.$$

If $\alpha = (i_1, \ldots, i_n) \in I_\infty$, let

$$W_\alpha = W_{i_1} \circ \ldots \circ W_{i_n}.$$

In the case $n = 0$ we interpret I^0 as the empty set and $W_\emptyset$ as the identity operator. If

$$\alpha = (i_1, \ldots, i_n) \quad \text{and} \quad \beta = (j_1, \ldots, j_m),$$

then

$$\alpha\beta = (i_1, \ldots, i_n, j_1, \ldots, j_m)$$

and we have

$$W_\alpha W_\beta = W_{\alpha\beta}.$$

The triple $(H, e, \{W_i: i \in I\})$ is said to be a *spectral representation* of a stochastic process iff

(i) for every $I' \subset I$ the operator $\sum_{i \in I'} W_i$ is a contraction on H;

(ii) $(W_\alpha e, e) \geqslant 0$ for every $\alpha \in I_\infty$;

(iii) $We = W^* e = e$.

It is known (see [5]) that for every spectral representation $(H, e, \{W_i: i \in I\})$ there exists a process (T, Q), $Q = \{Q_i: i \in I\}$, such that

$$m\left(\bigcap_{j=0}^{k} T^{-j} Q_{i_j}\right) = (W_\alpha e, e), \qquad \alpha = (i_0, \ldots, i_k) \in I_\infty.$$

A spectral representation $(H, e, \{W_i: i \in I\})$ is said to be *reduced* if

(iv) $H = \overline{\mathrm{Sp}}\{W_\alpha e: \alpha \in I_\infty\} = \overline{\mathrm{Sp}}\{W_\alpha^* e: \alpha \in I_\infty\}$.

It is known [5] that for every process (T, Q) there exists a reduced spectral representation.

Definition (cf. [2]). The process (T, Q) is said to be *finitary* if for every reduced spectral representation $(H, e, \{W_i: i \in I\})$ of (T, Q) the space H is finite dimensional.

It is proved in [2] (Corollary 2) that a process (T, Q) is finitary iff there exists a spectral representation $(H, e, \{W_i: i \in I\})$ with $\dim H < \infty$, and it follows that a functional of the finitary process is also finitary.

Theorem ([5]). *A reduced representation $(H, e, \{W_i: i \in I\})$ represents a Markov partition if and only if*

$$\dim(\mathrm{range}\, W_i) = 1 \quad \textit{for all } i \in I.$$

COROLLARY. *Markov processes are finitary.*

Proof. $\dim H \leqslant |I|$, where $(H, e, \{W_i: i \in I\})$ is a reduced representation of the Markov process (T, Q).

2. Example. Let l^2 denote the Hilbert space of all square summable sequences of complex numbers, equipped with the usual inner product $(\, , \,)$.

Let $e = (e_n)$, where $e_1 = 1$ and $e_n = 0$ for $n \geqslant 2$. We denote by π the projection on the first coordinate:

$$\pi x = x_1 = (x, e), \quad x \in l^2.$$

We consider the following two infinite matrices:

$$W_1 = [W_1^{i,j}], \quad W_2 = [W_2^{i,j}], \quad i, j \geqslant 1,$$

where

$$W_1^{1,1} = W_2^{1,1} = \tfrac{1}{2}, \quad W_1^{1,i} = W_1^{i,1} = a^{i-1}, \quad W_2^{1,i} = W_2^{i,1} = -a^{i-1},$$
$$W_1^{i,i} = a^{i-1}, \quad W_2^{i,i} = -a^{i-1}, \quad a \in (0, \tfrac{1}{10}), \ i \geqslant 2,$$

$$W_1^{i,j} = W_2^{i,j} = 0 \quad \text{for the remaining indices } i, j \geqslant 1.$$

We use the same letters to denote the linear operators of l^2 induced by W_1 and W_2.

Let $W = W_1 + W_2$. We have of course $W^{1,1} = 1$ and $W^{i,j} = 0$ for $(i, j) \neq (1, 1)$. It is also clear that $Wx = \pi(x) \cdot e$, $x \in l^2$.

We put $I = \{1, 2\}$.

Properties of the operation W_i, $i \in I$.

(1) *W_i are self-adjoint contractions, $i \in I$.*

Proof. It is enough to check that W_i are contractions. But this follows at once from the choice of a:

$$\|W_i x\|_2^2 \leqslant \left(\frac{1}{4} + 3 \sum_{k=1}^{\infty} (a^2)^k\right) \|x\|_2^2 = \left(\frac{1}{4} + \frac{3a^2}{1-a^2}\right) \|x\|_2^2 \leqslant \|x\|_2^2, \quad x \in l^2.$$

The operator W is of course also a self-adjoint contraction.

Let us consider the following cone:

$$C = \{x \in l^2 \colon 2\max(\pi x, 0) \geqslant \|x\|_1\}.$$

It is clear that $e \in C$ and $\pi x \geqslant 0$, $x \in C$.

(2) $$W_i C \subset C, \quad i \in I.$$

Proof. Let $x \in C$ and $y^i = W_i x$, $i \in I$. Hence, by the choice of a and the inequality $\pi x = x_1 \geqslant 0$ we have

$$\begin{aligned}
2\max(\pi y^i, 0) - \|y^i\|_1 &= y_1^i - \sum_{k=2}^{\infty} |y_k^i| \\
&= \tfrac{1}{2}x_1 - (-1)^i \sum_{k=1}^{\infty} a^k x_{k+1} - \sum_{k=2}^{\infty} a^{k-1}|x_1 + x_k| \\
&\geqslant \tfrac{1}{2}x_1 - \sum_{k=1}^{\infty} a^k |x_{k+1}| - \sum_{k=2}^{\infty} a^{k-1}|x_1| - \sum_{k=2}^{\infty} a^{k-1}|x_k| \\
&= \left(\frac{1}{2} - \frac{a}{1-a}\right)x_1 - 2\sum_{k=1}^{\infty} a^k x_{k+1} \geqslant 2ax_1 - 2a\sum_{k=2}^{\infty}|x_k| \\
&= 2a\big(2\max(\pi x, 0) - \|x\|_1\big) \geqslant 0, \quad \text{i.e.,} \quad W_i x \in C,\ i \in I.
\end{aligned}$$

Since $e \in C$ and $\pi x = (x, e) \geqslant 0$ for $x \in C$, property (2) implies at once

$$(W_\alpha e, e) \geqslant 0, \quad \alpha \in I_\infty. \tag{3}$$

Now we want to check

$$(W_\alpha W W_\beta e, e) = (W_\alpha e, e)(W_\beta e, e), \quad \alpha, \beta \in I_\infty. \tag{4}$$

Proof. Let $U = W_\alpha$, $V = W_\beta$, $\alpha, \beta \in I_\infty$.
Since $Wx = \pi x \cdot e$ and $\pi x = (x, e)$, $x \in l^2$, we have

$$(UWVe, e) = \big(U(\pi(Ve)\cdot e), e\big) = (Ue, e)\pi(Ve) = (Ue, e)(Ve, e),$$

which gives the desired equality.
Let now

$$H = \overline{\mathrm{Sp}}\{W_\alpha e\colon \alpha \in I_\infty\}.$$

It follows from (1) that

$$H = \overline{\mathrm{Sp}}\{W_\alpha^* e\colon \alpha \in I_\infty\}.$$

We assert that

$$\dim H = \infty. \tag{5}$$

Proof. We consider the sequence $(V^n) \subset l^2$ such that

$$V_1^n = 0, \quad V_k^n = a^{(k-1)n}, \quad k \geqslant 2,\ n \geqslant 1.$$

Let us observe that $V^n \in H$ for every $n \geqslant 1$. Since $e \in H$ and $W_1 H \subset H$, we have

$$V^1 = W_1 e - \tfrac{1}{2}e \in H.$$

If $V^n \in H$ for some $n \geqslant 1$, we have

$$V^{n+1} = W_1(V^n) - \sum_{i=1}^{\infty} a^{ki} e \in H,$$

i.e.,

$$V^n \in H \quad \text{for every } n \geqslant 1.$$

The formula

$$\det \begin{bmatrix} a & a^2 & \dots & a^k \\ a^2 & a^4 & \dots & a^{2k} \\ \dots & \dots & \dots & \dots \\ a^k & a^{2k} & \dots & a^{k^2} \end{bmatrix} = a^{\binom{k+1}{2}} \cdot \prod_{1 \leqslant i < j \leqslant k} (a^i - a^j), \qquad k \geqslant 1,$$

implies that the sequence (V^n) is linearly independent. Therefore $\dim H = \infty$.

Now, from (1), (3) and the trivial equality $We = e$ it follows that the triple $(H, e, \{W_i \colon i \in I\})$ is a reduced spectral representation.

Therefore there exist a dynamical system $(X, \mathscr{B}, m, T)$ and a measurable partition $Q = \{Q_1, Q_2\}$ of X such that

$$m(Q_{i_0} \cap T^{-1} Q_{i_1} \cap \dots \cap T^{-k} Q_{i_k}) = (W_{i_0} \dots W_{i_k} e, e), \qquad i_0, \dots, i_k \in I, \ k \geqslant 1.$$

We claim that the process (T, Q) is weak Bernoulli but it is not finitary.

Let $n \geqslant 2$, $k \geqslant 0$ and let

$$A \in \bigvee_{i=0}^{k} T^{-i} Q \quad \text{and} \quad B \in \bigvee_{i=n+k}^{n+2k} T^{-i} Q$$

be arbitrary.

Let us put

$$A = Q_{i_0} \cap T^{-1} Q_{i_1} \cap \dots \cap T^{-k} Q_{i_k}, \qquad B = T^{-(n+k)} Q_{j_0} \cap \dots \cap T^{-(n+2k)} Q_{j_k}.$$

Since $W^n = W$ for every $n \geqslant 1$, by (4) we have

$$m(A \cap B) = (W_\alpha W^{n-1} W_\beta e, e) = (W_\alpha e, e)(W_\beta e, e) = m(A) m(B),$$

where

$$\alpha = (j_k, \dots, j_0), \ \beta = (i_k, \dots, i_0) \in I_\infty.$$

This means that the partitions $\bigvee_{i=0}^{k} T^{-i} Q$ and $\bigvee_{i=n+k}^{n+2k} T^{-i} Q$ are independent, and so (T, Q) is weak Bernoulli.

On the other hand, property (5) and Corollary 1 of [2] imply that (T, Q) is not finitary.

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