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THE GENERALIZED PURITY LAW FOR ERGODIC MEASURES: A SIMPLE PROOF

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1. Introduction and preliminaries

1.1. A finite positive Borel measure μ on $G = \mathbf{T}$ or $\mathbf{R}$ is *ergodic* with respect to some countable subgroup D, acting by translation on G, when

for every Borel set E which is invariant under the translations by elements of D, either $\mu(E) = 0$ or μ is concentrated on E.

We shall also say that μ is *D-ergodic*. G. Brown and W. Moran ([1], [2]) pointed out this property for infinite convolution products of discrete measures and the classical Jessen–Wintner Purity Law, which asserts that such measures are either discrete, or absolutely continuous, or continuous and purely singular, is an immediate consequence. They proved moreover the following strengthening (μ^n denotes the nth convolution power of μ):

GENERALIZED PURITY LAW. *Let μ be a finite positive Borel measure on $\mathbf{T}$ or $\mathbf{R}$. If μ is ergodic with respect to some countable subgroup, then either μ is discrete, or there exists a positive integer n such that μ^n is absolutely continuous, or μ^n is singular with respect to any translate of μ^p whenever $n \neq p$.*

The original proof of this result, which can also be found in [4] and [6], uses J. L. Taylor's critical point theory for the Gelfand spectrum of convolution measure algebras. We shall give here a reasonable and self-contained proof, using only basic results on duality for locally compact abelian groups and elementary arguments in Gelfand theory (the Shilov idempotent theorem). Let us state a simpler but equivalent result, which the first author had to refer to in a recent work in pure ergodic theory [5]:

THEOREM 1. *Let μ be a finite positive Borel measure on $\mathbf{T}$ or $\mathbf{R}$ which is ergodic with respect to some countable subgroup. If μ and its convolution square are not mutually singular, then μ is either discrete or absolutely continuous.*

The definition of D-ergodicity is straightforwardly extended to locally compact abelian groups G other than $\mathbf{T}$ and $\mathbf{R}$. We keep the assumption of metrizability, although it is not necessary, since it allows us to work with standard measure spaces and is natural in ergodic theory. The general statement is somewhat weaker:

THEOREM 2. *Let μ be a finite positive Borel measure on a metrizable l.c.a. group G which is ergodic with respect to some countable subgroup. If μ and its convolution square are not mutually singular, there exists a l.c.a. group H continuously embedded in G such that μ is absolutely continuous with respect to the Haar measure of H.*

1.2. Theorem 1 follows from Theorem 2 since, when $G = \mathbf{T}$ or $\mathbf{R}$, such a subgroup is either discrete or equal to G. This is classical, but we shall mention the argument (in Section 3).

We refer to [4] and [6] for the basic properties of D-ergodic measures, such as the fact that any convolution product of D-ergodic measures is still D-ergodic. Let τ_x denote the translation by $x \in G$. We do not require the quasi-invariance $\tau_d\mu \sim \mu$ for every $d \in D$ (the equivalence means mutual absolute continuity) since it does not hold for the simplest examples like infinite convolution products of discrete measures. Choosing any convex combination with positive coefficients of all the $\tau_d\mu$ ($d \in D$), we get a quasi-invariant measure ν and so a non-singular dynamical system (G, ν, D), defined up to equivalence, which is ergodic if and only if μ is D-ergodic. Besides, two quasi-invariant D-ergodic measures are either equivalent or mutually singular.

Therefore, in our three statements, we may assume that μ is quasi-invariant under the action of the subgroup D. Then it is immediate that Theorem 1 follows from the generalized purity law. Conversely, assume that μ is continuous and that μ^n and $\tau_x\mu^{n+k}$ are not mutually singular for some $n, k > 0$ and $x \in G$; they are then equivalent and we have $\mu^m \sim \tau_{px}\mu^{m+pk}$ for all $m \geq n$ and $p > 0$; letting $\nu = \tau_x\mu^k$, we have $\nu^p \sim \nu^{2p}$ for some p. As ν^p is still ergodic, Theorem 1 implies that ν^p is absolutely continuous and so is μ^{kp}.

So, we only have to prove Theorem 2.

1.3. Let us recall a few basic facts about (Polish) l.c.a. groups, which can be found in [4] and [10, Chaps. 1, 2]. The Haar measure of a l.c.a. group H is denoted by λ_H; if H_1 is a quotient l.c.a. group of H, a Borel set in H_1 has zero Haar measure if and only if its inverse image in H has zero Haar measure (the image of a finite absolutely continuous measure on H is absolutely continuous, and any finite absolutely continuous measure on H_1 is the image of an absolutely continuous measure on H).

The dual group $\widehat{H}$ of H is the group of all continuous characters of H, with the topology of uniform convergence on compact sets; if H is countable and therefore discrete, $\widehat{H}$ is compact and metrizable. If $\varphi : H_1 \to H_2$ is a continuous homomorphism of l.c.a. groups, the dual homomorphism $\widehat{\varphi} : \widehat{H}_1 \to \widehat{H}_2$ is one-to-one if and only if $\varphi(H_1)$ is dense.

The Fourier transform of a finite Borel measure ν on H is defined by $\hat{\nu}(\gamma) = \int \gamma \, d\nu$ for $\gamma \in \widehat{H}$.

We suppose henceforth that *μ is quasi-invariant and ergodic* under the action of D, and that $\mu * \mu \sim \mu$. By the D-ergodicity, μ is concentrated on a class modulo $\overline{D}$ and, from the hypothesis $\mu * \mu \sim \mu$, this class must be $\overline{D}$. So, we will assume that *D is dense.* Then the topology on G is Polish. G is written additively. As D is a dense subgroup, $\widehat{G}$ is continuously embedded as a dense subgroup in $\widehat{D}$.

2. Eigenvalues and characters. A non-zero function f in $L^\infty(\mu)$ is an *eigenfunction* for the action of D if for every $d \in D$, there exists a constant $\gamma(d)$ such that $f(x+d) = \gamma(d) f(x)$ μ-a.e. As a function of d, γ is a group character of D; by ergodicity, f has constant modulus and is determined up to a constant by γ. We shall say that γ is the *eigenvalue* corresponding to f. The set $e(\mu, D)$ of all eigenvalues is a subgroup of $\widehat{D}$ (we refer to [7] for a thorough study of such eigenvalue groups).

Besides, let $L(\mu)$ denote the space, isomorphic to $L^1(\mu)$, of all complex measures which are absolutely continuous with respect to μ. From the hypothesis $\mu * \mu \ll \mu$, $L(\mu)$ is closed under convolutions and thus is a Banach algebra under the total variation norm. A character of $L(\mu)$ is a non-zero multiplicative linear functional on $L(\mu)$, i.e. a non-zero function f in $L^\infty(\mu)$ such that, for all ν, $\nu' \in L(\mu)$,

$$\int f \, d\nu * \nu' = \int f(x+y) \, d\nu(x) d\nu'(y) = \int f(x) \, d\nu(x) \int f(y) \, d\nu'(y)$$

and this is equivalent to the functional equation (generalized character property):

$$f(x+y) = f(x) f(y) \qquad \mu \times \mu\text{–a.e.} \tag{1}$$

We denote by $\Delta(\mu)$ the set of all characters of $L(\mu)$. The equation (1) shows that, as a subset of the unit ball in $L^\infty(\mu)$, it is closed under complex conjugation and pointwise multiplication. Since the action of D commutes with convolutions, any character f of $L(\mu)$ is also an eigenfunction. In particular $|f|$ is constant and, by (1), $|f| = 1$ μ-a.e. So $\Delta(\mu)$ is a subgroup of the group of all unit modulus functions in $L^\infty(\mu)$.

The Gelfand topology on $\Delta(\mu)$ is nothing but the weak* topology and is locally compact ($\Delta(\mu) \cup \{0\}$ is compact). On the group of unit modulus functions, the weak* topology inherited from $L^\infty(\mu)$ is also the Polish group topology defined by the $L^2(\mu)$ metric since, when $|f| = |f_0| = 1$ μ-a.e.,

$$\int |f - f_0|^2 \, d\mu = 2\Big(\|\mu\| - \mathrm{Re} \int f \bar{f}_0 \, d\mu\Big).$$

Therefore $\Delta(\mu)$ is a Polish l.c.a. group. Of course, any continuous group character is a character of $L(\mu)$ and we have a continuous embedding of the dual group $\widehat{G}$ in $\Delta(\mu)$. The following lemma is not really needed for our proof but will allow us to identify $\Delta(\mu)$ and $e(\mu, D)$.

LEMMA 1. *For each eigenvalue $\gamma \in e(\mu, D)$, there is one and only one eigenfunction f_γ with eigenvalue γ which is a character of $L(\mu)$. This defines a group isomorphism from $e(\mu, D)$ onto $\Delta(\mu)$ and so a Polish l.c.a. group topology on $e(\mu, D)$.*

As the mapping which assigns to each character the corresponding eigenvalue is clearly a group homomorphism from $\Delta(\mu)$ to $e(\mu, D)$, it will be enough to prove the first assertion. Let f be any eigenfunction, with eigenvalue $\gamma \in \widehat{D}$. As $\mu * \mu \sim \mu$ we have, for each $d \in D$, $f(x+d) = \gamma(d)f(x)$ $\mu * \mu$-a.e., that is, from the definition of $\mu * \mu$ as the image of $\mu \times \mu$ by the mapping $(x, y) \to x + y$,

$$f(x + y + d) = \gamma(d) f(x + y) \qquad \mu \times \mu\text{-a.e.}$$

For μ-almost all y, $x \to f(x + y)$ is an eigenfunction with the same eigenvalue, and thus there exists a constant $g(y)$ such that $f(x + y) = g(y)f(x)$ μ-a.e.(y). The function g so defined is clearly measurable, so that $g(y)f(x) = f(x + y) = g(x)f(y)$ $\mu \times \mu$-a.e., whence $g(x) = cf(x)$ μ-a.e., where c is a non-zero constant. So,

$$f(x + y) = cf(x)f(y) \qquad \mu \times \mu\text{-a.e.}$$

and $f_\gamma = cf$ is a character. Also, c is the only non-zero constant such that cf satisfies (1).

The topology on $e(\mu, D)$ will be defined by the mapping $\gamma \to f_\gamma$ and the weak* topology of $L^\infty(\mu)$ (or equivalently the $L^2(\mu)$ topology). Given a continuous group character $\gamma \in \widehat{G}$, we still denote by γ its restriction to D which is an element of $e(\mu, D)$, namely the eigenvalue corresponding to the eigenfunction $\gamma(x)$. Then $f_\gamma(x) = \gamma(x)$ μ-a.e.

3. A special case. We consider here the case when the l.c.a. group $e(\mu, D) = \Delta(\mu)$ is exactly the dual group $\widehat{G}$ of G. Then the Gelfand transform of a measure in $L(\mu)$ is merely its Fourier transform.

In the following lemma, we could make use of the structure theorem for l.c.a. groups, but it will be enough to refer to the following simpler and well-known result ([10], Theorem 2.3.2):

Any l.c.a. group which contains a dense homomorphic image of $\mathbf{Z}$ *is either compact or isomorphic to* $\mathbf{Z}$.

Such a group is called *monothetic*. Observe that, by duality, the property above is equivalent to the fact that any l.c.a. group continuously embedded in $\mathbf{T}$ is either discrete or equal to $\mathbf{T}$ (and the analogue for $\mathbf{R}$ is immediate by taking a quotient in $\mathbf{T}$).

LEMMA 2. *Assume moreover* $e(\mu, D) = \widehat{G}$. *Then* μ *is absolutely continuous.*

Suppose first that G contains a compact open subgroup K. The Haar measure λ_K of K is absolutely continuous and its Fourier transform is the characteristic function of the annihilator $\{\gamma \in \widehat{G};\ \gamma(x) = 1 \text{ for all } x \in K\}$ of K. This set is compact and open in $\widehat{G}$, which is the Gelfand spectrum of $L(\mu)$. By the Shilov idempotent theorem [3] there exists an element ν of $L(\mu)$ with $\hat{\nu} = \widehat{\lambda_K}$. By the unicity of the Fourier transform, $\lambda_K = \nu$ and thus $\lambda_K \ll \mu$. Since μ is ergodic, it is absolutely continuous with respect to a combination of translates of λ_K, i.e. with respect to λ_G.

In the general case, we use a modification of the construction given in [10] (Lemma 2.4.2) in order to get a quotient l.c.a. group which contains a compact open subgroup. Let V be a symmetric compact neighborhood of 0 in G. As D is dense, we can find finitely many elements $x_1, \ldots, x_n$ of D such that $V + V$ is contained in $\bigcup(V + x_i)$. Then we choose a subfamily $x_{j_1}, \ldots, x_{j_k}$ so that the subgroup D_0 it spans is discrete in the topology of G, and which is maximal for this property.

Let $H = G/D_0$, and let W and $y_1, \ldots, y_n$ be the images of V and $x_1, \ldots, x_n$ in H. By the maximality assumption, none of the closed groups K_i generated by the y_i may be isomorphic to $\mathbf{Z}$; as those groups are monothetic, they are compact and so is the sum $K = W + K_1 + \ldots + K_n$. By the construction $W + W \subset K$ and it follows that $K + K \subset K$; moreover, K is symmetric and so it is a subgroup of H. As $K \supset W$ and W is a neighborhood of 0 in H, K is an open subgroup.

Let π be the natural projection of G onto H and let ν be the image of μ by π. Clearly, $\nu * \nu \ll \nu$, ν is quasi-invariant and ergodic under the action of D/D_0. f is an eigenvalue for $(H, \nu, D/D_0)$ (or a character of $L(\nu)$) if and only if the D_0-invariant function $f \circ \pi$ is an eigenvalue for (G, μ, D) (or a character of $L(\mu)$). Thus $e(\nu, D/D_0) = \{\gamma \in \widehat{G};\ \gamma(d) = 1 \text{ for all } d \in D_0\} = \widehat{H}$, and both have the topology induced by the topology of $\widehat{G}$. By the first part of the proof $\nu \ll \lambda_H$.

Now, for every λ_G-null set N in G, $\lambda_G(N + D_0) = 0$ and it follows that $\pi(N)$ is a λ_H-null set, so that $\mu(N + D_0) = \nu(\pi(N)) = 0$ and thus $\mu(N) = 0$. Therefore $\mu \ll \lambda_G$.

4. Proof of Theorem 2. Let H be the dual group of $e(\mu, D)$. We write H additively, we denote by $\langle x, \gamma\rangle$ the duality. Since $\widehat{G} \subset e(\mu, D)$, $e(\mu, D)$ is a dense subgroup of $\widehat{D}$ and we have an isomorphism of D with a dense subgroup of H; we shall consider D to be still embedded in H. By duality, there is also a continuous homomorphism φ from H into G such that

$$(2) \qquad \langle \varphi(x), \gamma\rangle = \langle x, \gamma\rangle \qquad \text{for all } x \in H \text{ and } \gamma \in \widehat{G}.$$

We shall construct a Borel cross section μ-a.e. for φ, for which the image measure of μ is absolutely continuous.

Let E be a countable dense subgroup of $e(\mu, D)$ such that $E \cap \widehat{G}$ is dense in $\widehat{G}$, and let $\widehat{E}$ be its dual group. For μ-almost all x, $f_{\gamma\gamma'}(x) = f_\gamma(x) f_{\gamma'}(x)$ holds for every γ, $\gamma' \in E$, so that we have a mapping h from G to $\widehat{E}$ with, μ-almost everywhere,

$$(3) \qquad \langle h(x), \gamma\rangle = f_\gamma(x) \qquad \text{for every } \gamma \in E.$$

This mapping is clearly Borel, from the definition of the topology on $\widehat{E}$.

Let ν be the image of μ by h. The Fourier transform of ν is given on E by

$$\hat{\nu}(\gamma) = \int \langle h(x), \gamma\rangle \, d\mu(x) = \int f_\gamma(x) \, d\mu(x);$$

so, $\hat{\nu}$ can be extended to a continuous function on $e(\mu, D)$, and this function is clearly still positive definite. Note that H is a subgroup of $\widehat{E}$. By the Bochner theorem, ν is concentrated cn H.

Thus $h(x) \in H$ μ-a.e. Moreover, for every $\gamma \in E \cap \widehat{G}$, (2) and (3) yield

$$\langle \varphi \circ h(x), \gamma\rangle = \langle h(x), \gamma\rangle = f_\gamma(x) = \langle x, \gamma\rangle \qquad \mu\text{-a.e.},$$

and, as $E \cap \widehat{G}$ is dense in $\widehat{G}$, this proves $\varphi \circ h(x) = x$ μ-a.e.

So, h is one-to-one from a subset of full μ-measure in G to H, and μ is the image of ν by φ. From $f_\gamma(x + y) = f_\gamma(x) f_\gamma(y)$ $\mu \times \mu$-a.e. for all $\gamma \in E$, we get

$$(4) \qquad h(x + y) = h(x) + h(y) \qquad \mu \times \mu\text{-a.e.},$$

and, for every $d \in D$, as $f_\gamma(x + d) = \gamma(d) f_\gamma(x) = \langle h(x) + d, \gamma\rangle$ μ-a.e. for all $\gamma \in E$,

$$(5) \qquad h(x + d) = h(x) + d \qquad \mu\text{-a.e.}$$

By (4), $\nu * \nu$ is the image of $\mu * \mu$ by h and thus $\nu * \nu \ll \nu$. By (5), ν is quasi-invariant under D and the system (H, ν, D) is isomorphic to (G, μ, D); it is ergodic and $e(\nu, D) = e(\mu, D) = \widehat{H}$. So, (H, ν, D) satisfies the hypothesis of Lemma 2, and ν is absolutely continuous.

Now, we may consider the quotient l.c.a. group $H_1 = H/\ker\varphi$ to be continuously embedded in G. Then, as μ is the image of ν by φ, it is absolutely continuous with respect to the Haar measure of H_1. This ends the proof of Theorem 2.

5. Remarks

1. We finally conclude that $L(\mu)$ is the group algebra $L^1(H_1)$, whose Gelfand spectrum is the dual group $\widehat{H}_1$; it follows that $H_1 \simeq H$ and φ is one-to-one. This implies that $\widehat{G}$ is dense in $e(\mu, D)$, but we cannot prove this fact directly.

2. Theorem 1 yields an alternative proof for a result of V. Mandrekar and M. Nadkarni [9]:

If μ is a positive Borel measure on $\mathbf{T}$ or $\mathbf{R}$ such that $\tau_x\mu \sim \mu$ μ-a.e., then μ is either discrete or absolutely continuous.

Indeed, $\mu * \mu \sim \mu$ follows then from $\mu * \mu = \int \tau_x \mu \, d\mu$ and it is easy to show that μ is ergodic with respect to any countable subgroup of $H(\mu) = \{x;\ \tau_x\mu \sim \mu\}$ which is dense in the Polish topology defined by the operation of $H(\mu)$ on $L(\mu)$ (see [7]).

Conversely, we cannot deduce Theorem 1 from the result above, since we would have to prove a priori that the hypothesis $\mu * \mu \sim \mu$ implies $\tau_x\mu \sim \mu$ μ-a.e.; this is a consequence of Theorem 1, but we can construct a non-ergodic μ with $\mu * \mu \sim \mu$ and $\tau_x\mu \perp \mu$ for every $x \neq 0$.

3. In [8], we give a further strengthening of the generalized purity law, corresponding to the stronger version of Theorem 1, for a positive D-ergodic measure μ on $\mathbf{T}$ or $\mathbf{R}$:

If μ is continuous and singular, then there exists a Borel set E such that μ is concentrated on E and $\mu * \mu(E - x) = 0$ for every x.

This could also be proved without reference to J. L. Taylor's theory (with noticeable complications).

In the same note, we show that the ergodicity condition may be replaced by the weaker purity assumption:

for every Borel set E, either $\mu(E) = 0$ or μ is concentrated on a countable union of translates of E.

Under this hypothesis we are not able to give a direct proof.

4. The main argument here is that $\Delta(\mu)$ is a l.c.a. group, which allows the construction in Section 4. This can be done more generally, for a positive finite Borel measure μ on G with $\mu * \mu \ll \mu$ (so that $L(\mu)$ is a convolution algebra), when the group of modulus one functions in $\Delta(\mu)$ is open in $\Delta(\mu)$.

Then, in the language of J. L. Taylor [11], 1 is critical in $\Delta(\mu)$ and his main result is that μ is not singular with respect to the Haar measure of some continuously embedded l.c.a. group. In our proof, we need the additional assumption of ergodicity since, in Lemma 2, when G does not contain a compact open subgroup, μ must be quasi-invariant under the action of the discrete group D_0, in order to obtain a proper quotient algebra of $L(\mu)$ whose spectrum is the desired subset of $\Delta(\mu)$.

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