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ON SEMI-INNER PRODUCT SPACES, I

BY

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1. Introduction. In his paper [6] Lumer, for the first time introduced the concept of semi-inner product spaces. A complex (real) vector space X is called a *complex (real) semi-inner product space* if corresponding to arbitrary pair of elements $x, y \in X$, there is defined a complex (real) number $[x, y]$ which satisfies the following properties, for $x, y, z \in X$, λ complex (real):

(i) $[x+y, z] = [x, z]+[y, z]$, $\quad [\lambda x, y] = \lambda[x, y]$,

(ii) $[x, x] > 0 \quad$ for $x \neq 0$ (strict positivity),

(iii) $|[x, y]|^2 \leqslant [x, x]\,[y, y]$.

Henceforth semi-inner product would be abbreviated to s. i. p. With the setting $\|x\| = [x, x]^{\frac{1}{2}}$, every s. i. p. space becomes a normed linear space, and an s. i. p. space, more generally a normed space, is an *inner product space* if and only if the norm satisfies the parallelogram law:

$$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2.$$

Lumer further studied the relationship between the approximate point spectrum and the closed numerical range. The numerical range of a bounded operator T, defined on an s. i. p. space X, is the set of all complex numbers $\{[Tx, x]: x \in X, \|x\| = 1\}$.

Berkson [1] studied some properties of the s. i. p. spaces and Bade functionals. Giles [3] with the idea of obtaining the analogues of the Riesz representation theorem, orthogonality relation as studied by James [4], and Gâteaux differentiability of the norm, was led to introduce the concept of continuity of s. i. p. spaces which plays a very important role in all his theorems and so would do in our theorems. Giles also needed many other axioms than originally enunciated by Lumer. The techniques adopted by Giles are standard in the theory of Hilbert space.

Our main interest in this paper is to study the analogues of the Riesz

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representation theorem and the unique decomposition theorem. In obtaining the analogue of the Riesz representation theorem, we replace the uniform convexity [2] and the homogenity property $[x, \lambda y] = \bar{\lambda}[x, y]$ by the inequality $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$ where $0 < \mu < 1$ [5]. It is easy to see that every normed space, satisfying the inequality $\|x+y\|^2+\mu^2\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2$, is uniformly convex. It may be mentioned here that a normed linear space satisfying this inequality with $\mu = 1$ is an inner product space (Schoenberg [8]). Thus the inequality just mentioned is stronger than uniform convexity as used by Giles. But then it would follow that the normed spaces whose norms satisfy the inequality have richer structure than that of a uniformly convex normed space.

Throughout μ is taken to be positive and less than one.

2. In this section, we establish the unique decomposition theorem among other results of the s. i. p. spaces. For some additional results, we require some additional structure on the s. i. p. space.

Definition 2.1. A normed vector space is *strictly convex* if whenever $\|x\|+\|y\| = \|x+y\|$, where $x, y \neq 0$, then $y = \lambda x$ for some real $\lambda > 0$.

Berkson characterized strict convexity of the norm in terms of the s. i. p. properties. We quote his result in the form of the following

LEMMA 2.2. *An s. i. p. space is strictly convex if and only if, whenever,* $[x, y] = \|x\|\,\|y\|$, *where* $x, y \neq 0$, *then* $y = \lambda x$ *for some real* $\lambda > 0$.

Definition 2.3. A sequence x_n of elements of the s. i. p. space X is said to *converge weakly* in the first argument to an element $x \in X$ if $[x_n, y] \to [x, y]$ for every $y \in X$. The weak convergence with respect to the second argument is defined in an analogous manner.

PROPOSITION 2.4. *In an s. i. p. space* X, *the strong convergence implies weak convergence in the first argument and the weak limit is unique. In the case of weak convergence with respect to the second argument, the weak limit is unique if the s. i. p. space is strictly convex.*

Proof. The first part of the proposition is easy and hence we omit it. As regards the second part we proceed as follows. Let y be the weak limit of a sequence $\{y_n\}$ in X with respect to the second argument. Let y' be another weak limit of the sequence $\{y_n\}$. Then $[x, y] = [x, y']$ for each $x \in X$. Putting $x = y$ in $[x, y] = [x, y']$, we have $[y, y] = [y, y']$ or $\|y\|^2 \leqslant \|y\|\,\|y'\|$, that is $\|y\| \leqslant \|y'\|$. Putting $x = y'$ in $[x, y] = [x, y']$, by the similar arguments as above, we have $\|y'\| \leqslant \|y\|$. Hence $\|y\| = \|y'\|$. Putting $x = y$ in $[x, y] = [x, y']$ and using $\|y\| = \|y'\|$, we have $\|y\|\,\|y'\| = [y, y']$. Hence, by an appeal to Lemma 2.2, we have $y = y'$.

Definition 2.5. An s. i. p. space is said to be a *continuous* s. i. p. space if $\mathrm{Re}\{[y, x+\lambda y]\} \to \mathrm{Re}[y, x]$ for all real $\lambda \to 0$.

We shall now introduce the concept of orthogonality due to Giles [3] and James [4]. The following definition is due to Giles.

Definition 2.6. Let X be an s. i. p. space. We say x is *orthogonal* to y (written as $x \perp y$) if $[y, x] = 0$, where $x, y \in X$.

It is to be noted that this relation is not symmetric, that is if $x \perp y$, then y is not necessarily orthogonal to x.

James defined orthogonality in a different way:

Definition 2.7. Let X be a normed linear space. We say $x \perp y$ if $\|x + \lambda y\| \geqslant \|x\|$ for any scalar λ, where $x, y \in X$.

Under certain conditions, as shown by Giles, Definitions 2.6 and 2.7 coincide. We state the connection in the form of

LEMMA 2.8. *In a continuous s. i. p. space X, $x \perp y$ if and only if $\|x + \lambda y\| \geqslant \|x\|$ for any scalar λ, where $x, y \in X$.*

We prove the following

THEOREM 2.9. *Let X be a complete and continuous s. i. p. space which satisfies the inequality*

$$\|u+v\|^2 + \mu^2 \|u-v\|^2 \leqslant 2\|u\|^2 + 2\|v\|^2, \tag{2.9.1}$$

then, for every closed proper subspace N of X, there is a non-zero vector orthogonal to N and any $x \in X$ can be expressed in the form $x = y + z$, where y belongs to N and z is orthogonal to N. Moreover, this representation is unique.

Proof. Let N be a closed vector subspace of X, and let $x \in X \sim N$. Let $d = d(x, N)$ denote the distance between x and N, where $d(x, N) = \inf\{\|y - x\|, y \in N\}$. Now there exists a sequence $\{y_n\} \in N$ such that $\lim \|y_n - x\| = d$. Now proceeding, as in [7], p. 71, using inequality (2.9.1) instead of the usual parallelogram law, we have $\|x - y_0\| = \inf\{\|x - y\|, y \in N\}$, where y_0 is the limit of the sequence $\{y_n\}$. Putting $x - y_0 = z$, we have $\|z\| \leqslant \|y_0 - y + z\| = \|y' + z\|$, where $y' = y_0 - y \in N$. Hence, by Definition 2.7, $z \perp N$, and $x = y_0 + z$. The uniqueness of the decomposition is a matter of routine arguments.

In the next theorem, we obtain a more general result than Theorem 4 of [6], p. 32, provided the s. i. p. space has some additional structure present in it.

THEOREM 2.10. *In a complete, continuous s. i. p. space X satisfying the inequality $\|x+y\|^2 + \mu^2\|x-y\|^2 \leqslant 2\|x\|^2 + 2\|y\|^2$,*

$$\sigma(T) \subset \overline{W(T)},$$

where $\sigma(T)$ and $\overline{W(T)}$ denote, respectively, the spectrum and the closure of the numerical range of a bounded operator T defined on X.

Proof. Let λ be a complex number such that $\lambda \notin \overline{W(T)}$. Then let us write

$$0 < d = d(\lambda, \overline{W(T)}) = \inf_{u \in \overline{W(T)}} |u-\lambda| \quad \text{or} \quad d \leqslant |[Tx, x]-\lambda|$$
$$= |[Tx, x]-\lambda[x, x]| = |[Tx-\lambda x, x]|,$$

where $x \in X$ and $\|x\| = 1$. This can be written as

$$d\|x\|^2 \leqslant |[Tx-\lambda x, x]| \leqslant \|Tx-\lambda x\| \, \|x\|$$

or

$$d\|x\| \leqslant \|(T-\lambda)x\| \quad \text{or} \quad d\|(T-\lambda)^{-1}y\| \leqslant \|y\|$$

for all y in the range of $(T-\lambda)$. Thus $(T-\lambda)^{-1}$ is a bounded operator on the range of $T-\lambda$. $\overline{R(T-\lambda)} \neq X$, for otherwise $R(T-\lambda)$, is dense in X. Since $(T-\lambda)$ is invertible on $R(T-\lambda)$, it follows that $(T-\lambda)^{-1}$ exists on X. Hence $\lambda \in \varrho(T)$ (the resolvent set of T). Now if $\overline{R(T-\lambda)} \neq X$, then, by decomposition theorem, there exists an $x_0 \neq 0$ such that $[(T-\lambda)x, x_0] = 0$ for $x \in X$. In particular, $[Tx_0, x_0] = \lambda[x_0, x_0]$. By Lemma 2.8, without loss of generality, we can choose $\|x_0\| = 1$, hence $[Tx_0, x_0] = \lambda$, therefore $\lambda \in W(T)$, which is a contradiction.

3. In this section we propose to discuss the analogue of the Riesz representation theorem for s. i. p. spaces.

Definition 3.1. The norm of a normed space is said to be *Gâteaux differentiable* if, for all $x, y \in X$ and real λ,

$$\lim_{\lambda \to 0} \frac{\|y+\lambda x\| - \|y\|}{\lambda}$$

exists.

The following Lemma due to Giles [4] shows the relationship between the s. i. p. space and the Gâteaux differentiability of the norm.

LEMMA 3.2. *In a continuous s. i. p. space X, the norm is Gâteaux differentiable, and*

$$\lim_{\lambda \to 0} \frac{\|y+\lambda x\| - \|y\|}{\lambda} = \frac{\operatorname{Re}\{[x, y]\}}{\|y\|}, \quad \text{where } x, y \in X.$$

THEOREM 3.3. *In a continuous s. i. p. space X, which is complete with respect to its norm and in addition the norm satisfies the inequality*

$$\|u+v\|^2 + \mu^2\|u-v\|^2 \leqslant 2\|u\|^2 + 2\|v\|^2, \tag{3.3.1}$$

every continuous linear functional f, defined on X, can be represented by $f(x) = [x, y]$, where y is unique.

Proof. We shall divide the proof in two parts: one for the real case and the other for the complex case.

Real case: Let f be a continuous linear functional defined on X. We may, without any loss of generality, suppose that $\|f\| = 1$. Let us choose a sequence $y_n \in X$ such that $\|y_n\| = 1$ and $|f(y_n)| \to \|f\| = 1$. We may suppose (after multiplying by a suitable complex number of modulus 1) that $f(y_n) = |f(y_n)|$. Thus we have $f(y_n) \leqslant 1, f(y_n) \to 1$. Let ε be chosen so that $0 < \varepsilon < 1$, then, for sufficiently large $n, m, f(y_n) > 1-\varepsilon$ and $2-2\varepsilon \leqslant f(y_n+y_m) \leqslant \|y_n+y_m\|$. By inequality (3.3.1), we have

$$\mu^2\|y_n-y_m\|^2 \leqslant 2\|y_n\|^2+2\|y_m\|^2-\|y_n+y_m\|^2 \leqslant 4-(2-2\varepsilon)^2.$$

Hence, $\{y_n\}$ is a Cauchy sequence and converges to y, say. Then $\|y_n\| \to \|y\| = 1$. It is also clear that y is unique. Now, for $\lambda > 0$, we have $f(y \pm \lambda x) \leqslant \|y \pm \lambda x\|$ and $f(y) = \|y\|$ (since $\|f\| = 1$, by supposition). Hence

$$f(x) = \frac{f(y+\lambda x)-f(y)}{\lambda} \leqslant \frac{\|y+\lambda x\|-\|y\|}{\lambda},$$

$$f(x) = -\frac{f(y-\lambda x)-f(y)}{\lambda} \geqslant -\frac{\|y-\lambda x\|-\|y\|}{\lambda}.$$

Therefore as $\lambda \to 0$, we have, by Lemma 3.2, $f(x) = \mathrm{Re}\{[x, y]\} = [x, y]$, since the space under consideration is a real space.

Complex case: Following the arguments in Lumer ([6], Theorem 2, p. 31), let us write $f(x) = f_1(x) - if_1(ix)$, where f_1 is the functional defined over the real s. i. p. space which is complete with respect to its norm and is also continuous. Obviously, $\|f_1\| \leqslant \|f\| = 1$ (suppose). Since $f_1(y) = \|y\|$, we have $\|f_1\| = 1$. Now, as in the real case, $f_1(y+\lambda x) \leqslant \|y+\lambda x\|$. Hence, arguing as in the real case, $f_1(x) = \mathrm{Re}\{[x, y]\}$. Again $f_1(ix) = \mathrm{Re}\{[ix, y]\} = -\mathrm{Im}[x, y]$. Hence $f(x) = f_1(x) - if_1(ix) = \mathrm{Re}\{[x, y]\} + i\ \mathrm{Im}[x, y] = [x, y]$. This completes the proof of the theorem.

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