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MAPPING ARCWISE CONNECTED CONTINUA ONTO CYCLIC CONTINUA

BY

W. KUPERBERG (HOUSTON)

1. By a *compactum* we mean* a compact metric space and a connected compactum will be called a *continuum*. The term *mapping* will be used for a continuous function. We say that a mapping $f\colon X\to Y$ is *inessential* if it is homotopic to a constant mapping, otherwise we call it *essential*. The unit circle in the plane will be denoted by S^1; we shall consider S^1 as the set of all complex numbers with the absolute value 1. For a space X and a point x_0 fixed in X, the set of all homotopy classes of mappings of the pair $(S^1, 1)$ into the pair (X, x_0) has the well-known group structure; the group is denoted by $\pi_1(X, x_0)$ or just $\pi_1(X)$ if X is arcwise connected, and it is called the *fundamental group* of X. On the other hand, the set $[X, S^1]$ of all homotopy classes of mappings of X into S^1 has then natural Abelian group structure, induced by the group structure of S^1, as it was noticed by Bruschlinsky [2].

Recall that for any mapping $f\colon X\to Y$ there are homomorphisms $f_*\colon \pi_1(X)\to\pi_1(Y)$ and $f^*\colon \pi^1(Y)\to\pi^1(X)$ that are said to be *induced* by f. These homomorphisms are defined by

$$f^*[\alpha] = [f\circ\alpha] \text{ for any mapping } \alpha\colon (S^1, 1)\to(X, x_0),$$

and

$$f^*[\beta] = [\beta\circ f] \text{ for any mapping } \beta\colon Y\to S^1,$$

where the square brackets indicate the homotopy class of the mapping. In both groups $\pi_1(X)$ and $\pi^1(X)$, the neutral element is the homotopy class consisting of all inessential mappings.

For any arcwise connected space X, let $A(X)$ denote the subset of the group $\pi^1(X)$, consisting of the homotopy classes of all mappings $\beta\colon X\to S^1$ such that $\beta_*\big(\pi_1(X)\big)\approx 0$. One can easily prove the following propositions:

PROPOSITION 1. *For any acrwise connected space X, $A(X)$ is a subgroup of $\pi^1(X)$.*

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PROPOSITION 2. *If f is a mapping of an arcwise connected space X into an arcwise connected space Y, then $f^*(A(Y)) \subset A(X)$.*

These propositions tell us that A *is a (contravariant) functor from the category of arcwise connected spaces to the category of Abelian groups.* Indeed, for any arcwise connected space X, $A(X)$ is an Abelian group, and for any mapping f as in Proposition 2, we can define the homomorphism $f^A\colon A(Y) \to A(X)$ induced by f just restricting both the domain and the range of the homomorphism $f^*\colon \pi^1(Y) \to \pi^1(X)$. Observe that the functor A has the homotopy invariance property, since the cohomotopy functor has this property. This means that if two mappings $f, g\colon X \to Y$ are homotopic, then $f^A = g^A$.

PROPOSITION 3. *If X is a locally connected continuum, then $A(X) \approx 0$.*

Proof. Let $p\colon R^1 \to S^1$ be the well-known universal covering mapping; R^1 denotes the set of real numbers and $p(t) = e^{2\pi i t}$ for any $t \in R$. If $[\alpha] \in A(X)$, i.e. $\alpha\colon X \to S^1$ is a mapping such that $\alpha_*(\pi_1(X)) \approx 0$, then there is a lifting of α, that is a mapping $\tilde{\alpha}\colon X \to R^1$ such that $p \circ \tilde{\alpha} = \alpha$ (see [4], Chapter 2), since each locally connected continuum is locally arcwise connected (see [3]). But R^1 is contractible which makes $\tilde{\alpha}$ inessential. Therefore α is inessential, $[\alpha] = 0$.

Proposition 3 tells us that the functor A is most useful when applied to non-locally-connected continua. The main result of this paper is the following

THEOREM. *Let X and Y be arcwise connected continua. If f is a mapping of X onto Y, then f^A is a monomorphism.*

Note that Proposition 3 can be obtained as a simple corollary from this theorem, since each locally connected continuum is a continuous image of the segment $I = [0, 1]$ and $A(I) \approx 0$.

2. We shall prove the theorem formulated in Section 1 using the following equivalent expression of it:

(*) *Let X and Y be arcwise connected continua and let f be a mapping of X onto Y. If $g\colon Y \to S^1$ is an essential mapping such that $g_*(\pi_1(Y)) \approx 0$, then $g \circ f\colon X \to S^1$ is essential.*

Proof. Since Y is a compactum, we can assume that Y is a subset of the Hilbert cube Q. Let T denote the product $Q \times S^1$ and let $G \subset T$ be the graph of the mapping g,

$$G = \{(y, s) \in Q \times S^1;\ y \in Y,\ s = g(y)\}.$$

The mapping $\gamma\colon Y \to T$ defined by $\gamma(y) = (y, g(y))$ is a homeomorphic embedding, $\gamma(Y) = G$. Further, let $p\colon T \to S^1$ be the projection, $p(q, s) = s$ for any $(q, s) \in T$. Observe that for any mapping $z\colon Z \to Y$, the composition $g \circ z$ is essential if and only if the composition $\gamma \circ z$ is essential, since $g = p \circ \gamma$ and p is a homotopy equivalence by contractibility of the Hilbert cube Q. In particular, γ is essential since so is g, and therefore the inclusion $i\colon G \to T$ is an essential mapping. Moreover, $i_*(\pi_1(G)) \approx 0$ since $g_*(\pi_1(Y)) \approx 0$. Now, as assumed in (*), let f be a mapping of an arcwise connected continuum X onto Y and suppose, on the contrary, that $g \circ f$ is inessential. Then $f_1 = \gamma \circ f$ is also inessential. Consider the covering mapping $\varphi\colon \tilde{T} \to T$, where $\tilde{T}$ is the product $R \times Q$ of the real line R and the Hilbert cube Q, and φ is the "wrapping function" $\varphi(t, q) = (e^{2\pi i t}, q)$. For any mapping $z\colon Z \to T$, a mapping $\tilde{z}\colon Z \to \tilde{T}$ is said to be a *lifting* of z if $\varphi \circ \tilde{z} = z$. Observe that z has a lifting if and only if z is inessential. Indeed, if z has a lifting, then it is inessential since $\tilde{T}$ is contractible. On the other hand, if z is inessential, i. e. there is a homotopy $z_t\colon Z \to T$ $(0 \leqslant t \leqslant 1)$ such that $z_0 = z$ and z_1 is a constant mapping, then z_1 can be trivially lifted and, by the homotopy lifting property of covering spaces (see [4], Chapter 2), the entire homotopy z_t can be lifted. In particular, there is a lifting $\tilde{z}_0$ of $z_0 = z$. Thus, f_1 has a lifting $\tilde{f}_1\colon X \to \tilde{T}$, but the inclusion $i\colon G \to T$ has no lifting. The image $\tilde{f}_1(X)$ is contained in the set $\tilde{G} = \varphi^{-1}(G)$ and, since X is arcwise connected, the image $\tilde{f}_1(X)$ is actually contained in a path-component P of the set $\tilde{G}$. Let us notice the following properties of P:

(i) *φ maps P onto G*, since G is acrwise connected and each path in G can be lifted.

(ii) *φ maps P one-to-one into T*, otherwise there would be a loop in G whose lifting would be a non-loop, which would yield $i_*(\pi_1(G)) \not\approx 0$.

(iii) *P is non-compact.* Otherwise, by (i) and (ii) the restriction $\varphi|_P$ would be a homeomorphism of P onto G and $(\varphi|_P)^{-1}$ would define a lifting of the inclusion $i\colon G \to T$, which is impossible.

In particular, it follows from (iii) that $\tilde{f}_1(X)$ is a proper subset of P. Further, by (ii), $f(X) = \varphi(\tilde{f}_1(X))$ is a proper subset of G and, therefore, $f(X)$ is a proper subset of Y, a contradiction which completes the proof.

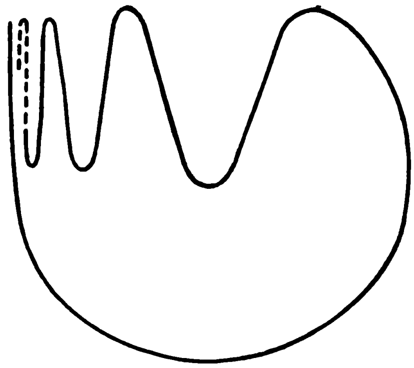
3. COROLLARY 1. *If a continuum Y is a continuous image of an arcwise connected continuum X, then the group $A(X)$ contains a subgroup isomorphic to the group $A(Y)$.*

COROLLARY 2. *If X and Y are arcwise connected continua, $A(X) \approx 0$ and $A(Y) \not\approx 0$, then there is no continuous mapping of X onto Y.*

A continuum is said to be *cyclic* if it can be mapped essentially onto S^1. Otherwise it is called *acyclic.*

COROLLARY 3. *If X is an acyclic arcwise connected continuum and Y is a cyclic continuum with $\pi_1(Y) \approx 0$, then there is no mapping of X onto Y.*

Example. The planar curve known as the "Warsaw circle" (sketched on the figure) is not a continuous image of any arcwise connected acyclic continuum.



This example gives a negative answer to the following question concerning dendroids, i. e. arcwise connected and hereditarily unicoherent continua, asked by Michael A. Laidacker:

Is it true that each arcwise connected continuum is a continuous image of a dendroid?

Considering this question, A. Lelek conjectured that the "Warsaw circle" is not a continuous image of any dendroid and even of any arcwise connected acyclic continuum, which appears to be true.

COROLLARY 4. *If a continuum Y is a continuous image of the cone over a compactum X, then $A(Y) \approx 0$.*

Indeed, the cone CX over a compactum X is arcwise connected and $A(CX) \approx 0$ since CX is contractible. By Corollary 2 we get $A(Y) \approx 0$.

Let us remark that Corollary 4 is a generalization of Proposition 3, since the segment $I = [0, 1]$ is the cone over a single point.

COROLLARY 5. *If Y is an arcwise connected continuum with $A(Y) \not\approx 0$, then for any compactum X, each mapping of X onto Y is essential.*

Proof. Consider X as the base of the cone CX. If a mapping f of X onto Y were inessential, then there would exist an extension $\bar{f}\colon CX \to Y$ of f. The mapping $\bar{f}$ would be onto, which would contradict Corollary 4.

PROBLEM. Does there exist an arcwise connected continuum X such that, for any arcwise connected continuum Y, $A(X)$ contains an isomorphic copy of $A(Y)$? (**P 905**)

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HOUSTON, HOUSTON, TEXAS

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ON 1-DIMENSIONAL CONTINUA WITHOUT THE FIXED-POINT PROPERTY

BY

JOHN R. MARTIN (SASKATOON, SASKATCHEWAN)

The following two problems were communicated to the author by Professor Lloyd Tucker.

PROBLEM 1. Does there exist a 1-dimensional continuum X without the fixed-point property such that every retract of X has the fixed-point property with respect to one-to-one maps?

PROBLEM 2. In problem 1 replace "one-to-one" by "onto".

The purpose of this note* is to provide examples which answer the two above questions in the affirmative, and to pose new questions by placing additional restrictions on the continua in problems 1 and 2.

Example 1. Our first example is essentially the same as an example given by Young [5], p. 884. Let C_1 be a continuum in the right half xy-plane joining the point $(0,3)$ to the interval $I_1 = [-3, -1]$ of the y-axis, C_1 being homeomorphic to the closure of the graph of $y = \sin(1/x)$, $0 < x \leqslant \pi$, with I_1 corresponding to the limiting interval of the graph. Let $C_2(I_2)$ be the image of $C_1(I_1)$ under the rotation of the xy-plane about the origin 0 through an angle of π. Let $T = T_1 \cup T_2 \cup T_3$ be a triod consisting of the subintervals T_1, T_2 on the y-axis joining the origin 0 to $(0,-1)$, respectively $(0,1)$, and an arc T_3 which joins 0 to $p = (0,4)$ and whose interior lies below the xy-plane. Let A be a set lying in the xy-plane homeomorphic to a half-open interval such that A

(1) has only its endpoint p in common with $C_1 \cup C_2 \cup T$ and

(2) "converges" to $C_1 \cup C_2$ in such a way that

(a) there is a sequence of arcs $S_1, S_2, S_3, \ldots$ filling up A such that $S_i \cap S_j = \emptyset$ for $j \neq i-1, i+1$, and $S_i \cap S_j$ is an end point of S_i and S_j for $j = i-1, i+1$, and

(b) $C_1 = \lim S_{2j-1}$, $C_2 = \lim S_{2j}$.

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Define $X_1 = C_1 \cup C_2 \cup T \cup A$. Then X_1 is a 1-dimensional (indeed, rational) continuum. We define a fixed-point free map $f\colon X_1 \to X_1$ which is a composition of two discontinuous functions f_1 and f_2. Let $f_1\colon X_1 \to X_1$ be such that on $C_1 \cup C_2 \cup T_1 \cup T_2$ it is a rotation in the xy-plane about 0 through an angle of π, and is the identity otherwise. Let $f_2\colon X_1 \to X_1$ be a function that is a homeomorphism on A such that for each i, S_i is mapped on S_{i+1}, that is the identity on $C_1 \cup C_2$, that maps T_j, $j = 1, 2$, homeomorphically onto $T_j \cup T_3$, and maps T_3 homeomorphically onto S_1. Then $f = f_2 f_1$ is continuous and fixed point free.

Now any one-to-one map $h\colon X_1 \to X_1$ must preserve triods. Since 0 is the only "triple point" in X_1, we must have $h(0) = 0$ and therefore X_1 has the fixed-point property with respect to one-to-one maps. To complete the example we show that every proper retract of X_1 has the fixed-point property. Since X_1 is an arcwise connected continuum, every retract of X_1 is an arcwise connected continuum. Moreover, no retract of X_1 in $C_1 \cup \cup C_2 \cup T$ can contain a neighborhood N of I_1 (or I_2). For otherwise, a subinterval of A would necessarily be retracted onto the non-locally connected space N which is impossible. Consequently, the only proper retracts of X_1 are singleton points, arcs, or triods, all of which are absolute retracts for compact metric spaces and hence have the fixed-point property [3].

Example 2. Our second example X_2 is the subspace obtained from X_1 by removing the interior of the triod T from X_1, i. e., $X_2 = C_1 \cup C_2 \cup A$. Then the restriction $f|X_2$ of f to X_2 is a fixed-point free map from X_2 into X_2.

First we show that X_2 has the fixed-point property with respect to onto maps. Let $g\colon X_2 \to X_2$ be an onto map. Since path components must be preserved under g, it follows that $g(A) = A$. For otherwise, $g(A) \subset C_1 \cup \cup C_2$ and hence $g(X_2) \subset C_1 \cup C_2$. Let h be a homeomorphism from the half-open interval $[0, 1)$ onto A, and define a map H from A into the real numbers $\boldsymbol{R}$ by

$$H(x) = h^{-1}\big(g(x)\big) - h^{-1}(x) \quad \text{for each } x \text{ in } A.$$

Let q be a point in A such that $h(q) = p$, where p is the initial endpoint of A, and let D denote the subinterval of A with endpoints p and q. Since $H(q) < 0$ and $H(p) > 0$, there is a point d in D such that $H(d) = 0$. Then $h^{-1}\big(g(d)\big) = h^{-1}(d)$ and hence $g(d) = d$. Therefore X_2 and every subspace homeomorphic to it has the fixed-point property with respect to onto maps.

Using an argument similar to that used in Example 1, we can easily show that all the retracts of X_2 which are not homeomorphic to X_2 are either singleton points or arcs and thus have the fixed-point property.

PROBLEM 1′. Is there an example for Problem 1 which is (a) planar? (b) planar and arcwise connected? (**P 906**)

PROBLEM 2'. Is there an example for Problem 2 which is (a) arcwise connected? (b) planar and arcwise connected? (**P 907**)

Remarks. 1. Of course, no example can contain a simple closed curve as a retract. In particular, no 1-dimensional example X can be locally connected. For since X does not have the fixed-point property, it cannot be a dendrite [2]. Hence it contains a simple closed curve C. Since X is 1-dimensional and locally connected, it follows that X is not unicoherent about C and, therefore, C is a retract of X ([4], p. 216).

2. Even if we drop the condition of 1-dimensionality, no planar example X can be locally connected. For X must separate the plane $\boldsymbol{R}^2$, since every locally connected non-separating plane continuum has the fixed-point property [1]. Let p be a point in one of the bounded complementary domains of X. Then X must contain a simple closed curve, say J, such that J separates $\boldsymbol{R}^2$, and p lies in Int J ([4], p. 107). Since all simple closed curves in $\boldsymbol{R}^2$ are plane equivalent and the unit circle S^1 is a retract of $\boldsymbol{R}^2\setminus\{0\}$, it follows that J is a retract of $\boldsymbol{R}^2\setminus\{p\}$ and hence is a retract of X.

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THE UNIVERSITY OF SASKATCHEWAN
SASKATOON, CANADA

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