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ON COMMUTATORS ON NILPOTENT LIE GROUP

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Let X be a connected, simply connected, nilpotent Lie group. Knapp and Stein defined in [2] a generalization of the Euclidean singular integral of the Mihlin-Calderon-Zygmund type on X. We assume that the reader is familiar with [2].

Let $\Omega\colon X \to C$ be a C^∞-function on $X\setminus\{1\}$, where 1 is the identity of X. Denote by σ_r $(r > 0)$ a one-parameter group of dilations of X and suppose that

$$\Omega(\sigma_r x) = \Omega(x) \quad \text{for every } r > 0 \text{ and every } x \in X. \tag{1}$$

A *norm function* on X is a C^∞-function $|x|$ from $X\setminus\{1\}$ to the positive real numbers, having the following properties:

(a) $|x^{-1}| = |x|$,

(b) $|\sigma_r x| = r^q|x|$ for a fixed number $q > 0$,

(c) the measure $|x|^{-1}dx$ is invariant under dilations.

We refer to [1] and [2] for various examples of X, σ_r, and $|x|$. It was proved in [2] that, under the condition

$$\int\limits_{c<|x|<d} \Omega(x)\,dx = 0 \quad \text{for some } c \text{ and } d \text{ with } 0 < c < d,$$

for any $f \in L^2(X)$ the limit

$$Tf(x) = \lim_{\substack{\varepsilon\to 0\\ M\to\infty}} \int\limits_{\varepsilon<|y|<M} |y|^{-1}\,\Omega(y)f(yx)\,dy$$

exists in $L^2(X)$, and $f \to Tf$ is a bounded operator in $L^2(X)$. More precisely, for every integer k the operator

$$T_k f(x) = \int\limits_{2^{k-1}<|y|<2^k} |y|^{-1}\,\Omega(y)f(yx)\,dy$$

is defined and

$$Tf = \sum_{-\infty}^{\infty} T_k f, \quad f \in L^2(X).$$

We denote by L_a the operator of multiplication by a bounded continuous function a on X. If $X = \boldsymbol{R}^n$, $a(x)$ is smooth, and $\lim_{|x|\to\infty} a(x) = 0$, then it is well known that the commutator

$$[L_a, T] = L_a T - T L_a$$

is a compact operator.

In fact, this is true for far more general variable coefficient singular integral operators (pseudo-differential operators) with compact supports. Applying the method developed by Knapp and Stein we shall now prove the same result in the context of the nilpotent Lie group X. First we recall the definition of the Hölder condition.

We say that a function $h\colon X \to \boldsymbol{C}$ satisfies the *Hölder condition* if there exist $M > 0$ and a $(0 < a \leqslant 1)$ such that, for any $x, y \in X$,

$$|h(x) - h(y)| \leqslant M \|xy^{-1}\|^a,$$

where $\|\cdot\|$ denotes the Euclidean norm on X.

Now we are ready to formulate

THEOREM. *Let a be a bounded continuous function on X which satisfies the Hölder condition. Assume that*

$$\lim_{|x|\to\infty} a(x) = c,$$

where $|\cdot|$ denotes the norm function. Then the commutator $[T, L_a]$ is a compact operator in $L^2(X)$.

Note first that we can assume $c = 0$. Moreover, without loss of generality we can also take a with compact support. In fact, since $c = 0$, we can choose a sequence a_s such that

$$\|a - a_s\|_\infty \to 0 \text{ as } s \to \infty \quad \text{and} \quad \operatorname{supp} a_s \subset K(1, R_s),\ R_s > 0.$$

Thus we have

$$[L_a, T] = [L_a - L_{a_s}, T] + [L_{a_s}, T],$$

and this equality justifies our assumption.

In the proof of the Theorem we shall use the following

PROPOSITION. *If the function $a\colon X \to \boldsymbol{C}$ satisfies the Hölder condition and has a compact support, then for*

$$T_k f(x) = \int_{2^{k-1} \leqslant |y| < 2^k} |y|^{-1} \Omega(y) f(yx)\, dy \quad (k = 0, -1, -2, \ldots)$$

the following inequality holds:

$$(*)\qquad \|[T_k, L_a]f\|_2^2 \leqslant C\cdot 2^{k\beta}\|f\|^2, \quad \textit{where } \beta > 0 \textit{ and } f \in L^2(X).$$

Proof. Put

$$\Omega_k(y) = \begin{cases} \Omega(y) & \text{if } 2^{k-1} \leqslant |y| < 2^k, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in L^2(X)$ we have

$$\|[T_k, L_a]f\|_2^2 \leqslant \int\Big[\int |y|^{-1}|\Omega_k(y)|\cdot|a(yx)-a(x)|\cdot|f(yx)|\,dy\Big]^2 dx.$$

Now

$$(2)\qquad |a(yx)-a(x)| \leqslant M\|yx\cdot x^{-1}\|^a = M\|y\|^a.$$

But

$$(3)\qquad \|y\| \leqslant M_1|y|^d,$$

as was proved in [2], p. 498. Thus, by the Young inequality and (2), (3), we have

$$\begin{aligned}
&\int\Big[\int |y|^{-1}|\Omega_k(y)|\cdot|a(yx)-a(x)|\cdot|f(yx)|\,dy\Big]^2 dx \\
&\quad\leqslant MM_1^a\cdot 2^{adk}\int\Big[\int |y|^{-1}|\Omega_k(y)|\cdot|f(yx)|\,dy\Big]^2 dx \\
&\quad= MM_1^a\cdot 2^{adk}\big\||y|^{-1}|\Omega_k(y)|*f\big\|_2^2 \leqslant MM_1^a\cdot 2^{adk}\big\||y|^{-1}|\Omega_k(y)|\big\|_1^2\|f\|_2^2 \\
&\quad\leqslant \sup_{|y|\leqslant 1}|\Omega_k(y)|^2 MM_1^a\cdot 2^{adk}\Big(\int_{2^{k-1}\leqslant|y|<2^k}|y|^{-1}dy\Big)^2\|f\|_2^2.
\end{aligned}$$

By Proposition 2 of [2] the integral

$$\int_{2^{k-1}\leqslant|y|<2^k}|y|^{-1}dy = N$$

is independent of k, and so $(*)$ holds with

$$\beta = ad \quad\text{and}\quad C = \sup_{|y|\leqslant 1}|\Omega(y)|MM_1^aN^2.$$

Thus the proof is complete.

Proof of the Theorem. We have

$$\begin{aligned}
[L_a, T]f(x) &= \int_{|y|\leqslant 1}|y|^{-1}\Omega(y)[a(x)-a(yx)]f(yx)\,dy + \\
&\quad+\int_{|y|>1}|y|^{-1}\Omega(y)[a(x)-a(yx)]f(yx)\,dy = S_1f(x)+S_2f(x).
\end{aligned}$$

Note that the kernel $K(x, y)$ of the operator S_2 is square integrable. In fact, putting $yx = w$, we obtain

$$K(x, w) = |wx^{-1}|^{-1} \Omega(wx^{-1})[a(x)-a(w)].$$

But $\operatorname{supp} a \subset K(1, R)$ for a certain $R > 0$ and

$$\int |\Omega(x)|^2 h(|x|)\,dx = c(\Omega) \int_0^{\infty} h(n)\,dn$$

for any measurable h (see [2], p. 496). Consequently, we have

$$\iint |K(x, w)|^2 dx dw = \iint_{|wx^{-1}|>1} |K(x, w)|^2 dx dw$$
$$= \int_{|t|>1} |t|^{-2} |\Omega(t)|^2 dt \left(\int |a(x)|^2 dx + \int |a(tx)|^2 dx\right) < \infty.$$

Therefore S_2 is compact.

Now we show that S_1 is also compact. By [2] we obtain

$$S_1 = \left[L_a, \sum_{k=-\infty}^{0} T_k\right]$$

(strong convergence). On the other hand, by the Proposition, for any $m < n < 0$ we have

$$\left|\left|\left|\left[L_a, \sum_{m}^{n} T_k\right]\right|\right|\right| \leqslant \sum_{m}^{n} |||[L_a, T_k]||| \leqslant C^{1/2} \sum_{m}^{n} 2^{k\beta/2}.$$

Thus $\left[L_a, \sum_{n}^{0} T_k\right]$ satisfies the Cauchy condition and must be convergent in norm to S_1. Since obviously $\left[L_a, \sum_{n}^{0} T_k\right]$ are compact operators, so is S_1. Thus the proof is complete.

From the Theorem we derive the following

COROLLARY. *Assume that a satisfies the Hölder condition and has compact support. If b is a bounded continuous function such that*

$$\operatorname{supp} a \cap \operatorname{supp} b = \emptyset,$$

then the operator $L_a T L_b$ is compact.

This follows immediately from the equality

$$[L_a, T] L_b = L_a T L_b.$$

Remark 1. The Theorem remains true for $\Omega: X \to L(C^n)$ with entries Ω_{ij} such that $\Omega_{ij}(\sigma_r x) = \Omega_{ij}(x)$ and $a: X \to L(C^n)$ of the form $\tilde{a}(x) = a(x) I$, where I is the identity matrix.

Remark 2. In the case $X = \boldsymbol{R}^n$ we have $[T^*, T] = 0$, but for an arbitrary X this is not true in general. Moreover, if $S = [T^*, T] \neq 0$, then S is even not compact (because S is a translation invariant operator).

Remark 3. (**P 1272**) It seems that the Theorem should be true for a more general class of functions (or a different class). For example: is it sufficient to assume that $a \in C^\infty(X)$ and $\sup_{|y|<1} |a(yx) - a(x)| \to 0$ as $|x| \to \infty$? This is a sufficient condition in $\boldsymbol{R}^n$.

REFERENCES

[1] R. W. Goodman, *Nilpotent Lie groups*, Lecture Notes in Mathematics 562 (1976).

[2] A. W. Knapp and E. M. Stein, *Intertwining operators for semi-simple groups*, Annals of Mathematics 93 (1971), p. 489-578.

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