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APPLICATION OF RADEMACHER SYSTEMS TO OPERATOR CHARACTERIZATIONS OF BANACH LATTICES

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1. Notation and terminology. All Banach spaces and Banach lattices considered in this paper are assumed to be real.

For Banach spaces E, F and Banach lattices X, Y we denote by:

$\mathscr{L}(E, F)$ the Banach space of all (bounded linear) operators with the usual norm

$$\|T\|_{\mathscr{L}(E,F)} = \|T\| = \sup\{\|T(x)\|: \|x\| \leqslant 1\};$$

$\mathscr{F}(E, F)$ the closure in $\mathscr{L}(E, F)$ of all operators of finite rank;

$\mathscr{L}^r(X, Y)$ the Banach lattice of all regular operators with the norm

$$\|T\|_{\mathscr{L}^r(X,Y)} = \|T\|_r = \big\| |T| \big\|_{\mathscr{L}(X,Y)},$$

where $|T|$ is the modulus of T (Y is assumed here to be order complete);

$\mathscr{L}^{\mathrm{bo}}(E, Y)$ the Banach space of all bo-bounded operators (1) with the norm

$$\|T\|_{\mathscr{L}^{\mathrm{bo}}(E,Y)} = \|T\|_{\mathrm{bo}} = \inf\{\|y\|: |T(x)| \leqslant y \text{ whenever } \|x\| \leqslant 1\}$$

(following [8] or [12], this space would be denoted by $\prod(E, Y)$ or $\mathscr{B}_Y(E, Y)$, respectively);

$\mathscr{L}^l(X, F)$ the Banach space of all cone absolutely summing operators with the norm

$$\|T\|_{\mathscr{L}^l(X,F)} = \|T\|_l = \sup\Big\{\sum_{i=1}^n \|T(x_i)\|: \Big\|\sum_{i=1}^n |x_i|\Big\| \leqslant 1\Big\}.$$

The reader is referred to [13], Chapter IV, for details about the spaces $\mathscr{L}^r(X, Y)$ and $\mathscr{L}^l(X, F)$. Recall that the space $\mathscr{L}^l(X, F)$ was introduced by V. L. Levin who proved among other things that for each operator

(1) We recall that these operators map the unit ball of E into an order bounded subset of Y.

$T \in \mathscr{L}^{\mathrm{bo}}(E, Y)$ we have the adjoint operator $T^* \in \mathscr{L}^l(Y, E)$ and $\|T\|_{\mathrm{bo}} = \|T^*\|_l$ ([8], Lemmas 4 and 5; cf. also [13], Theorem IV.3.8).

We agree to identify a Banach lattice X with its canonical image in the second dual space X^{**}.

2. Formulation of the results and a lemma on Rademacher systems.

THEOREM A ([5], Theorem 1). *Let X and Y be Banach lattices and suppose $\mathscr{F}(X, Y) \subset \mathscr{L}^r(X, Y^{**})$. If, for some $p \in [1, \infty)$, Y (respectively, X^*) contains a vector sublattice order isomorphic to l^p, then X^* (respectively, Y) is order isomorphic to an AM-space.*

THEOREM B ([12], corollaire 2.3). *Let E be a Banach space and let Y be a Banach lattice with order continuous norm. If $\mathscr{L}(E, Y) = \mathscr{L}^{\mathrm{bo}}(E, Y)$, then either E or Y is finite dimensional.*

The proof of Theorem A given in [5] is based on a combinatorial result from [3] (about the existence of some special matrices) which is, in our opinion, off the point. The proof of Theorem B given in [12] is rather intricate.

The goal of this paper is to give unified proofs of these theorems by means of Rademacher systems in finite-dimensional Banach spaces. Tzafriri [14] seems to be the first who used this method for a similar purpose (cf. also Pełczyński and Singer [11]). Later Rademacher systems were applied by Ørno [10]. Our considerations are variations of those from [10].

Let a natural number n and a real number $p \in [1, \infty)$ be fixed. The (normalized) Rademacher system $r_1, \ldots, r_n$ in $l^p_{2^n}$ is defined as follows:

$$r_1 = \Big(\sum_{i=1}^{2^{n-1}} e_i - \sum_{i=2^{n-1}+1}^{2^n} e_i\Big)\Big/\Big\|\sum_{i=1}^{2^n} e_i\Big\|, \ldots, r_n = \Big(\sum_{i=1}^{2^n} (-1)^{i+1} e_i\Big)\Big/\Big\|\sum_{i=1}^{2^n} e_i\Big\|,$$

where $e_1, \ldots, e_{2^n}$ is the standard basis of $l^p_{2^n}$.

Our proofs of Theorems A and B are based on the following lemma which is implicit in [10].

LEMMA. *Let X be a Banach lattice and let $f_1, \ldots, f_n \in X^*_+$ be pairwise disjoint and $\|f_i\| \leqslant 1$. Define an operator $T\colon X \to l^p_{2^n}$ by*

$$T(x) = \sum_{i=1}^{n} f_i(x) r_i \quad \text{for } x \in X.$$

Then

$$\big\| |T| \big\| = \Big\| \sum_{i=1}^{n} f_i \Big\|, \tag{1}$$

$$\|T\| \leqslant A_p \Big\| \sum_{i=1}^{n} f_i \Big\|^{1/2}, \tag{2}$$

where A_p is a constant which is independent of n. In particular, if $\| |T| \| \leqslant K \|T\|$, then

$$\Big\| \sum_{i=1}^{n} f_i \Big\| \leqslant K^2 A_p^2.$$

Proof. From the disjointness of $f_1, \ldots, f_n$ we infer that

$$|T|(x) = \sum_{i=1}^{n} f_i(x) |r_i|,$$

and so (1) follows.

By the well-known Khintchine inequality, we have

$$\Big\| \sum_{i=1}^{n} f_i(x) r_i \Big\|_{l^p_{2^n}} \leqslant A_p \Big(\sum_{i=1}^{n} f_i^2(x) \Big)^{1/2}.$$

Hence

$$\|T\| \leqslant A_p \sup_{\|x\| \leqslant 1} \Big(\sum_{i=1}^{n} f_i^2(x) \Big)^{1/2} \leqslant A_p \sup_{\substack{i=1,\ldots,n \\ \|x\| \leqslant 1}} |f_i(x)|^{1/2} \sup_{\|x\| \leqslant 1} \Big| \sum_{i=1}^{n} f_i(x) \Big|^{1/2}$$

$$\leqslant A_p \Big\| \sum_{i=1}^{n} f_i \Big\|^{1/2}.$$

3. Proof of Theorem A. The unit ball of the space $\mathscr{F}(X, Y)$ equipped with the norm $\|\cdot\|_r$ is closed in the uniform topology, and so $\mathscr{F}(X, Y)$ is a Banach space under this norm (cf. [5] or [10]). Hence there exists a constant $K > 0$ such that for each $R \in \mathscr{F}(X, Y)$

$$\| |R| \| \leqslant K \|R\|. \tag{3}$$

(a) Let Y_1 be a vector sublattice of Y which is order isomorphic to l^p. For simplicity we identify Y_1 with l^p. The Lemma and (3) show that for all pairwise disjoint elements $f_1, \ldots, f_n \in X_+^*$ with $\|f_i\| \leqslant 1$ we have the inequality

$$\Big\| \sum_{i=1}^{n} f_i \Big\| \leqslant C, \tag{4}$$

where the constant C is independent of n and f_i. But, as was shown in [1], Theorem 5 (see also [5], Lemma 3), (4) is equivalent to the assertion that X^* is order isomorphic to an AM-space.

(b) Suppose $l^p \subset X^*$ and fix pairwise disjoint elements $y_1, \ldots, y_n \in Y_+$ with $\|y_i\| \leqslant 1$. Let $r_1, \ldots, r_n$ be the Rademacher system in $l^p_{2^n} \subset X^*$ and define $T \in \mathscr{F}(X, Y)$ by

$$T(x) = \sum_{i=1}^{n} r_i(x) y_i \qquad (x \in X).$$

Then

$$T^*(g) = \sum_{i=1}^{n} \langle y_i, g \rangle r_i \quad (g \in Y^*). \tag{5}$$

The inequality $|T^*| \leqslant |T|^*$ and (3) now yield

$$\| |T^*| \| \leqslant \| |T|^* \| = \| |T| \| \leqslant K \|T\| = K \|T^*\|.$$

Hence, in view of the Lemma, (4) holds with $C = K^2 A_p^2$ and f_i replaced by y_i, which completes the proof.

Remarks. (i) The proof given above shows that instead of the assumption $l^p \subset X^*$ (respectively, $l^p \subset Y$) we might only suppose that l^p is finitely lattice representable in X^* (respectively, Y), i.e. for each n there exist an n-dimensional vector sublattice Z_n in X^* (respectively, Y) and an order isomorphism T_n of Z_n onto l_n^p such that

$$\sup_n \|T_n\| \|T_n^{-1}\| < \infty.$$

(ii) It is worth-while to note that the single condition $\mathscr{F}(X, Y) \subset \mathscr{L}^r(X, Y^{**})$ (or even $\mathscr{L}(X, Y) = \mathscr{L}^r(X, Y)$) is not sufficient to draw the conclusion that X^* or Y is order isomorphic to an AM-space. The corresponding example was constructed in [2].

4. Proof of Theorem B. The scheme of Robert's demonstration [12] of the theorem is the following.

Suppose E is infinite dimensional. Then

(a) *Y is order isomorphic to $c_0(\Gamma)$ for some set Γ.*

(b) *The condition $\mathscr{L}(E, Y) = \mathscr{L}^{bo}(E, Y)$ then implies that Y is finite dimensional.*

It is the proof of (a) presented in [12] that is rather intricate, and we propose to give here a simpler one.

Note first that the assumption $\mathscr{L}(E, Y) = \mathscr{L}^{bo}(E, Y)$ implies the existence of a constant $K > 0$ such that $\|R\|_{bo} \leqslant K \|R\|$ for each $R \in \mathscr{L}(E, Y)$. Hence, in view of Levin's theorem mentioned at the end of Section 1,

$$\|R^*\|_l \leqslant K \|R^*\|. \tag{6}$$

As the norm of Y is order continuous by assumption, to prove (a) it is enough to show that Y is order isomorphic to an AM-space. The latter will be established as long as we verify condition (4) for Y. To this end fix pairwise disjoint elements $y_1, \ldots, y_n \in Y_+$ with $\|y_i\| \leqslant 1$. According to the well-known Dvoretzky theorem, there exists a 2^n-dimensional subspace $E_n \subset E^*$ which is close to $l_{2^n}^2$ (for example, the Banach-Mazur distance between E_n and $l_{2^n}^2$ is less than or equal to 2). For simplicity we identify E_n with l^2_{2n}. Let $r_1, \ldots, r_n$ be the Rademacher system in l^2_{2n}, and put, as in the proof of Theorem A,

$$T(x) = \sum_{i=1}^{n} r_i(x) y_i \qquad (x \in E).$$

Then $T^* \in \mathscr{L}(Y^*, l^2_{2n})$ and

$$\| |T^*| \|_{\mathscr{L}(Y^*, l^2_{2n})} \leqslant \|T^*\|_{\mathscr{L}^l(Y^*, l^2_{2n})} = \|T^*\|_{\mathscr{L}^l(Y^*, E^*)}.$$

Hence, in view of (6),

$$\| |T^*| \|_{\mathscr{L}(Y^*, l^2_{2n})} \leqslant K \|T^*\|_{\mathscr{L}(Y^*, l^2_{2n})},$$

and so an application of the Lemma completes the proof of (a).

For the sake of completeness we reproduce here Robert's proof of (b). By the theorem of Josefson [7] and Nissenzweig [9], there existst a sequence $\{f_n\} \subset E^*$ such that $\|f_n\| = 1$ and $f_n(x) \to 0$ for each $x \in E$. Suppose Γ is infinite and take a sequence $\{\gamma_n\}$ of its different elements. Put

$$T(x) = \sum_{n=1}^{\infty} f_n(x) e_n \qquad (x \in E),$$

where $e_n(\gamma) = 1$ if $\gamma = \gamma_n$ and $e_n(\gamma) = 0$ otherwise. It is evident that $T \in \mathscr{L}(E, c_0(\Gamma))$ but $\{T(x): \|x\| \leqslant 1\}$ is not order bounded in $c_0(\Gamma)$, a contradiction.

Remarks. (i) Still another proof of Theorem B has been given by Buhvalov [4].

(ii) In paper [6] by the second-named author there are some more results proved by using Rademacher systems. We mention here the following

THEOREM C. *Let X be a Banach lattice. The following two conditions are equivalent*:

(a) *X is order isomorphic to an AL-space*;

(b) *$\mathscr{F}(X, F) \subset \mathscr{L}^l(X, F)$ holds for some* (*every*) *infinite-dimensional Banach space F.*

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