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SPHERICAL FUNCTIONS ON FINITE SPLIT EXTENSIONS

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0. Introduction. If G is a finite group, let $\hat{G}$ denote the set of all irreducible complex characters of G. If $K \leqslant G$, $\chi \in \hat{G}$, and $\Psi \in \hat{K}$, then the *spherical function* $Y_{\chi\Psi}$ is defined by

$$Y_{\chi\Psi}(x) = |K|^{-1} \sum \{\chi(xy)\Psi(y^{-1}) : y \in K\}$$

for all $x \in G$. The nonzero spherical functions on G may be viewed as characters of the centralizer algebra $(CG)^K$, which consists of all elements in the complex group algebra CG that are constant on the orbits in G under conjugation by K. If

$$\chi | K = \sum \{c_{\chi\Psi} \Psi : \Psi \in \hat{K}\},$$

then $c_{\chi\Psi} = (\chi | K, \Psi)_K = Y_{\chi\Psi}(1)$ (these are the *restriction multiplicities*). If $c_{\chi\Psi}$ is 0 or 1 for every $\chi \in \hat{G}$ and $\Psi \in \hat{K}$, then K is said to be a *multiplicity free subgroup* of G.

Spherical functions on finite groups were first studied explicitly by Travis in [5], although certain spherical functions appeared implicitly in [2]. It seems that the only spherical functions (other than characters) on finite groups that have been explicitly calculated are those of the symmetric groups relative to the stabilizer of a point (see [1]). In the interest of making more examples available we compute herein the spherical functions relative to H and N for certain classes of split extensions $G = NH$ with $N \lhd G$.

Suppose G is a split extension of H by N, i.e. $N \lhd G$, $H \leqslant G$, $G = NH$, and $N \cap H = 1$. Then H acts as a permutation group on N with

$$h : n \to h^{-1}nh = n^h.$$

It will be convenient at times to write ${}^h n$ for $n^{(h^{-1})} = hnh^{-1}$, and n^{-h} for $(n^{-1})^h$. The subgroup H also acts as a permutation group on $\hat{N}$ with

$$\varphi^h(n) = \varphi(hnh^{-1}) = \varphi({}^h n).$$

We shall write H_φ for the stabilizer in H of φ for each $\varphi \in \hat{N}$.

A *Frobenius group* is a permutation group G acting transitively on a set A in such a way that the stabilizer of a point is nontrivial but only the identity fixes 2 or more points. Abstractly, G is a split extension NH, where H is the stabilizer of a point, $N \lhd G$, and $N \setminus \{1\}$ is the set of elements having no fixed points (N is the *Frobenius kernel* and H is a *Frobenius complement*).

The characters of a Frobenius group G are of two types (see [3], p. 94). If $\chi \in \hat{G}$, then either $\chi = \varphi^G$ for some $\varphi \in \hat{N}$, $\varphi \neq 1_N$, or else $\chi | N = 1_N$ and $\chi | H \in \hat{H}$. If φ, $\theta \in \hat{N}$, then $\varphi^G = \theta^G$ if and only if φ and θ are in the same H-orbit in $\hat{N}$. If $1_N \neq \varphi \in \hat{N}$, then $H_\varphi = 1$ ([3], p. 94).

I am indebted to the referee of an earlier version for pointing out an egregious error and for Theorem 6.1.

1. Spherical functions relative to N, G Frobenius.

We assume throughout this section and the next that $G = NH$ is a Frobenius group.

THEOREM 1.1. *Suppose $\chi = \varphi^G \in \hat{G}$, with $\varphi \in \hat{N}$, and $\theta \in \hat{N}$.*

(i) *If $\theta \notin \mathrm{Orb}_H \varphi$, then $Y_{\chi\theta} = 0$.*

(ii) *If $\theta \in \mathrm{Orb}_H \varphi$, then $Y_{\chi\theta} | (G \setminus N) \equiv 0$ and $Y_{\chi\theta} | N = (1/\theta(1)) \cdot \theta$.*

Proof. It is easily verified that

$$\varphi^G | (G \setminus N) \equiv 0 \quad \text{and} \quad \varphi^G | N = \sum \{\varphi^x : x \in H\}.$$

If $n \in N$ and $h \in H$, then

$$\begin{aligned} Y_{\chi\theta}(nh) &= |N|^{-1} \sum \{\chi(nhu)\theta(u^{-1}) : u \in N\} \\ &= |N|^{-1} \sum \{\chi(n \cdot {}^h u \cdot h)\theta(u^{-1}) : u \in N\} = 0 \end{aligned}$$

if $h \neq 1$. For $h = 1$ we have

$$\begin{aligned} Y_{\chi\theta}(n) &= |N|^{-1} \sum \{\varphi^G(nu)\theta(u^{-1}) : u \in N\} \\ &= |N|^{-1} \sum \Big\{ \sum \{\varphi^x(nu)\theta(u^{-1}) : u \in N\} : x \in H \Big\}. \end{aligned}$$

But $\sum \{\varphi^x(nu)\theta(u^{-1}) : u \in N\} = \varphi^x * \theta(n)$ is the (convolution) product of φ^x and θ in the group algebra CN, and if $\Psi \in \hat{N}$, then $(\Psi(1)/|N|) \cdot \Psi$ is a minimal central idempotent in CN. Thus $\varphi^x * \theta = 0$ if $\varphi^x \neq \theta$ and $\varphi^x * \theta = (|N|/\theta(1)) \cdot \theta$ if $\varphi^x = \theta$. In particular, if $\theta \notin \mathrm{Orb}_H \varphi$, then $Y_{\chi\theta} = 0$. If $\theta = \varphi^s$, $s \in H$, then $\varphi^x = \theta = \varphi^s$ if and only if $x = s$. Consequently,

$$Y_{\chi\varphi}(n) = |N|^{-1} \varphi^s * \theta(n) = (1/\theta(1)) \cdot \theta(n).$$

THEOREM 1.2. *Suppose $\chi \in \hat{G}$, with $\chi | H \in \hat{H}$, and $\theta \in \hat{N}$.*

(i) *If $\theta \neq 1_N$, then $Y_{\chi\theta} = 0$.*

(ii) *If $\theta = 1_N$, then $Y_{\chi\theta} = \chi$.*

Proof. If $n \in N$ and $h \in H$, then

$$\begin{aligned} Y_{\chi\theta}(nh) &= |N|^{-1} \sum \{\chi(nhu)\theta(u^{-1}) \colon u \in N\} \\ &= |N|^{-1} \sum \{\chi(n \cdot {}^h u \cdot h)\theta(u^{-1}) \colon u \in N\} \\ &= \chi(h)|N|^{-1} \sum \{\theta(u^{-1}) \colon u \in N\} \\ &= \chi(h)(1_N, \theta)_N = \begin{cases} 0 & \text{if } \theta \neq 1_N, \\ \chi(h) = \chi(nh) & \text{if } \theta = 1_N. \end{cases} \end{aligned}$$

THEOREM 1.3. *N is multiplicity free in G if and only if H is abelian.*

Proof. If $\chi = \varphi^G$ and $\theta \in \mathrm{Orb}_H \varphi$, then $c_{\chi\theta} = Y_{\chi\theta}(1) = 1$ by Theorem 1.1. If $\chi | H \in \hat{H}$ and $\theta = 1_N$, then $c_{\chi\theta} = Y_{\chi\theta}(1) = \chi(1) = 1$ for all such χ if and only if H is abelian. In all other cases $c_{\chi\Psi} = 0$.

2. Spherical functions relative to H, G Frobenius.

THEOREM 2.1. *Suppose $\chi \in \hat{G}$ and $\Psi \in \hat{H}$.*

(i) *If $\chi = \varphi^G$, with $\varphi \in \hat{N}$, $n \in N$, and $h \in H$, then*

$$Y_{\chi\Psi}(nh) = |H|^{-1} \varphi^G(n)\Psi(h).$$

(ii) *If $\chi | H = \Psi$, then $Y_{\chi\Psi} = (1/\chi(1)) \cdot \chi$.*

(iii) *If $\Psi \neq \chi | H \in \hat{H}$, then $Y_{\chi\Psi} = 0$.*

Proof. (i) If $x \in H$, then $\varphi^G(nhx) \neq 0$ if and only if $hx = 1$ or $x = h^{-1}$. Thus

$$Y_{\chi\Psi}(nh) = |H|^{-1} \sum \{\varphi^G(nhx)\Psi(x^{-1}) \colon x \in H\} = |H|^{-1}\varphi^G(n)\Psi(h).$$

(ii) and (iii). The proof is similar to that of Theorem 1.1:

$$\begin{aligned} Y_{\chi\Psi}(nh) &= |H|^{-1} \sum \{\chi(nhx)\Psi(x^{-1}) \colon x \in H\} \\ &= |H|^{-1} \sum \{\chi(hx)\Psi(x^{-1}) \colon x \in H\} \\ &= |H|^{-1} \chi * \Psi(h) = \begin{cases} \chi(nh)/\chi(1) & \text{if } \chi | H = \Psi, \\ 0 & \text{if } \Psi \neq \chi | H \in \hat{H}. \end{cases} \end{aligned}$$

THEOREM 2.2. *H is multiplicity free in G if and only if both N and H are abelian.*

Proof. If $\chi | H \in \hat{H}$, then $c_{\chi\Psi} = Y_{\chi\Psi}(1)$ is 0 or 1 in all cases. If $\chi = \varphi^G$ with $\varphi \in \hat{N}$, then

$$c_{\chi\Psi} = Y_{\chi\Psi}(1) = |H|^{-1}\varphi^G(1)\chi(1) = |H|^{-1}[G : N]\varphi(1)\chi(1) = \varphi(1)\chi(1),$$

which is 1 in all cases if and only if both N and H are abelian.

3. The characters of G, N abelian. In this section we present Mackey's construction, as in [4], for the characters of a finite split extension $G = NH$ with N abelian. This will establish notation for Sections 4 and 5.

Suppose $\varphi \in \hat{N}$ and T is an irreducible representation of H_φ with character η. Define T_φ on NH_φ by setting $T_\varphi(nh) = \varphi(n)T(h)$ for $n \in N$, $h \in H$. It is a routine matter to verify the following proposition:

PROPOSITION 3.1. *The function T_φ is an irreducible representation of NH_φ with character η_φ given by $\eta_\varphi(nh) = \varphi(n)\eta(h)$.*

In general, if $K \leqslant G$ and $\eta \in \hat{K}$, we write $\dot{\eta}$ for the function on G given by $\dot{\eta}|K = \eta$ and $\dot{\eta}|(G \setminus K) \equiv 0$.

PROPOSITION 3.2. *Suppose φ, $\theta \in \hat{N}$, $\eta \in \hat{H}_\varphi$, and $\Psi \in \hat{H}_\varphi$. Then*

(i) *η_φ^G is irreducible,*

(ii) *if $\theta \notin \mathrm{Orb}_H\varphi$, then $\eta_\varphi^G \neq \Psi_\theta^G$,*

(iii) *if $\theta = \varphi$ but $\eta \neq \Psi$, then $\eta_\varphi^G \neq \Psi_\varphi^G$.*

Proof. If $u, v \in N$ and $x, y \in H$, then

$$^{vy}(ux) = (vy)ux(vy)^{-1} = v \cdot {}^y u \cdot {}^{(yx)}v^{-1} \cdot {}^y x,$$

and so

$$\begin{aligned}
\eta_\varphi^G(ux) &= |NH_\varphi|^{-1} \sum \{\dot{\eta}_\varphi(^{vy}(ux)) \colon v \in N, y \in H\} \\
&= |NH_\varphi|^{-1} \sum \{\varphi(v)\varphi^y(u)\varphi^{yx}(v^{-1})\dot{\eta}^y(x) \colon v \in N, y \in H\} \\
&= |H_\varphi|^{-1} \sum \{\varphi^y(u)\dot{\eta}^y(x)(\varphi, \varphi^{yx})_N \colon y \in H\}.
\end{aligned}$$

By the Frobenius reciprocity theorem we obtain

$$\begin{aligned}
&(\eta_\varphi^G, \Psi_\theta^G) \\
&= (\eta_\varphi^G | NH_\theta, \Psi_\theta)_{NH_\theta} \\
&= |NH_\theta|^{-1} \sum \{\eta_\varphi^G(ux)\overline{\Psi_\theta(ux)} \colon u \in N, x \in H_\theta\} \\
&= |NH_\theta|^{-1}|H_\varphi|^{-1} \sum \{\varphi^y(u)\dot{\eta}^y(x)(\varphi, \varphi^{yx})_N\overline{\theta(u)\Psi(x)} \colon u \in N, x \in H_\theta, y \in H\} \\
&= |H_\theta|^{-1}|H_\varphi|^{-1} \sum \{(\varphi, \varphi^{yx})_N(\varphi^y, \theta)_N\dot{\eta}^y(x)\overline{\Psi(x)} \colon x \in H_\theta, y \in H\} \\
&= 0 \quad \text{if } \theta \notin \mathrm{Orb}_H\varphi,
\end{aligned}$$

proving (ii).

Now take $\theta = \varphi$. Then

$$(\eta_\varphi^G, \Psi_\varphi^G) = |H_\varphi|^{-2} \sum \{(\varphi, \varphi^{yx})_N(\varphi^y, \varphi)_N\dot{\eta}^y(x)\overline{\Psi(x)} \colon x \in H_\varphi, y \in H\}.$$

But $(\varphi^y, \varphi)_N = 0$ unless $y \in H_\varphi$, and if $y \in H_\varphi$, then

$$(\varphi, \varphi^{yx})_N = (\varphi^y, \varphi^{yx}) = (\varphi, \varphi^x) = (\varphi, \varphi) = 1 \quad \text{for } x \in H_\varphi.$$

Thus

$$\begin{aligned}
(\eta_\varphi^G, \Psi_\varphi^G) &= |H_\varphi|^{-2} \sum \{\eta^y(x)\overline{\Psi(x)} \colon x, y \in H_\varphi\} \\
&= |H_\varphi|^{-1} \sum \{\eta(x)\overline{\Psi(x)} \colon x \in H_\varphi\} = (\eta, \Psi)_{H_\varphi} = \delta_{\eta\Psi},
\end{aligned}$$

proving (i) and (iii).

Let $\varphi_1, \varphi_2, \ldots, \varphi_M$ be representatives of the H-orbits in $\hat{N}$. For each φ_i let $s(i) = c(H_{\varphi_i})$ be the class number of H_{φ_i}, let $\eta_{i1}, \eta_{i2}, \ldots, \eta_{is(i)}$ be all the irreducible characters of H_{φ_i}, and define $\chi_{ij} \in (NH_{\varphi_i})^\wedge$ by setting $\chi_{ij}(ux) = \varphi_i(u)\eta_{ij}(x)$ for $u \in N$, $x \in H_{\varphi_i}$, $1 \leqslant j \leqslant s(i)$. Thus all χ_{ij}^G are irreducible and they are distinct by Proposition 3.2.

THEOREM 3.1. $\hat{G} = \{\chi_{ij}^G\colon 1 \leqslant j \leqslant s(i),\ 1 \leqslant i \leqslant M\}$.

Proof. Note that $[G : NH_{\varphi_i}] = [H : H_{\varphi_i}]$, and hence

$$\chi_{ij}^G(1) = [H : H_{\varphi_i}]\Psi_{ij}(1).$$

Thus

$$\begin{aligned}\sum_{i,j} \chi_{ij}^G(1)^2 &= \sum_i \{[H : H_{\varphi_i}]^2 \sum_j \Psi_{ij}(1)^2\} \\ &= \sum_i [H : H_{\varphi_i}]^2 |H_{\varphi_i}| = \sum_i [H : H_{\varphi_i}]\cdot|H| \\ &= |H| \sum_i |\mathrm{Orb}_H \varphi_i| = |H|\cdot|N| = |G|.\end{aligned}$$

4. Spherical functions relative to N, N abelian.

THEOREM 4.1. *Suppose* $\chi = \eta_\varphi^G \in \hat{G}$ *and* $\theta \in \hat{N}$.

(i) *If* $\theta \notin \mathrm{Orb}_H\varphi$, *then* $Y_{\chi\theta} = 0$.

(ii) *If* $\theta = \varphi^s$, $s \in H$, *then*

$$Y_{\chi\theta}|NH_\theta = (\eta^s)_\theta \quad \textit{and} \quad Y_{\chi\theta}|(G\setminus NH_\theta) \equiv 0.$$

Proof. If $n, v \in N$ and $h \in H$, then $nhv = n\cdot{}^h v\cdot h$, and, as we saw in Section 3,

$$\chi(nhv) = |H_\varphi|^{-1} \sum \{\varphi^y(n)\varphi^{yh}(v)\dot{\eta}^y(h)(\varphi, \varphi^{yh})_N\colon y \in H\}.$$

Consequently,

$$\begin{aligned}Y_{\chi\theta}(nh) &= |N|^{-1} \sum \{\chi(nhv)\theta(v^{-1})\colon v \in N\} \\ &= |NH_\varphi|^{-1} \sum \{\varphi^y(n)\varphi^{yh}(v)\dot{\eta}^y(h)(\varphi, \varphi^{yh})\theta(v^{-1})\colon v \in N, y \in H\} \\ &= |H_\varphi|^{-1} \sum \{\varphi^y(n)(\varphi, \varphi^{yh})(\varphi^{yh}, \theta)\dot{\eta}^y(h)\colon y \in H\} \\ &= 0 \quad \text{if } \theta \notin \mathrm{Orb}_H\varphi.\end{aligned}$$

Suppose that $\theta = \varphi^s$, $s \in H$. Then

$$Y_{\chi\theta}(nh) = |H_\varphi|^{-1} \sum \{\varphi^y(n)(\varphi, \varphi^{yh})(\varphi^{yh}, \varphi^s)\dot{\eta}^y(h)\colon y \in H\},$$

and

$$(\varphi^{yh}, \varphi^s) = (\varphi^{yhs^{-1}}, \varphi) = 0 \quad \text{unless } y \in H_\varphi\cdot sh^{-1}.$$

If $y \in H_\varphi\cdot sh^{-1}$, write $y = h_1sh^{-1}$, $h_1 \in H_\varphi$, and note that then

$$(\varphi, \varphi^{yhy^{-1}}) = (\varphi, \varphi^{h_1sy^{-1}}) = (\varphi^y, \varphi^s) = (\varphi^{h_1sh^{-1}}, \varphi^s) = (\varphi, \varphi^{shs^{-1}}).$$

Thus

$$Y_{\chi\theta}(nh) = |H_\varphi|^{-1} \sum \{\varphi^y(n)(\varphi, \varphi^{sh})\dot{\eta}^y(h) \colon y \in H_\varphi \cdot sh^{-1}\}$$
$$= 0 \quad \text{unless } {}^sh \in H_\varphi \text{ or } h \in H_\varphi^s = H_{\varphi^s} = H_\theta.$$

If $h \in H_\theta$, then $H_\varphi \cdot sh^{-1} = H_\varphi \cdot s$, and so in that case

$$Y_{\chi\theta}(nh) = |H_\varphi|^{-1} \sum \{\varphi^y(n)\dot{\eta}^y(h) \colon y \in H_\varphi \cdot s\}$$
$$= |H_\varphi|^{-1} \sum \{\varphi^s(n)\dot{\eta}^y(h) \colon y \in H_\varphi \cdot s\}.$$

But if $y = h_2 s$, $h_2 \in H_\varphi$, then $\dot{\eta}^y = \dot{\eta}^{h_2 s} = \dot{\eta}^s$, and so

$$Y_{\chi\theta}(nh) = \varphi^s(n)\dot{\eta}^s(h) = \begin{cases} 0 & \text{if } h \notin H_{\varphi^s} = H_\theta, \\ \theta(n)\eta^s(h) = (\eta^s)_\theta(nh) & \text{if } h \in H_\theta. \end{cases}$$

THEOREM 4.2. *N is multiplicity free in G if and only if H is abelian.*

Proof. Taking $\varphi = 1_N$ we have $H_\varphi = H$ and $\chi = \eta_\varphi \in \hat{G}$ for each $\eta \in \hat{H}$. Then $c_{\chi\varphi} = Y_{\chi\varphi}(1) = \eta(1)$, so H must be abelian in order that N be multiplicity free. The converse is clear.

5. Spherical functions relative to H, N abelian. We continue to assume that G is a split extension NH with N abelian.

THEOREM 5.1. *Suppose $\varphi \in \hat{N}$, $\eta \in \hat{H}_\varphi$, $\chi = \eta_\varphi^G \in \hat{G}$, and $\Psi \in \hat{H}$. If $n \in N$ and $h \in H$, then*

$$(*) \qquad Y_{\chi\Psi}(nh) = |H|^{-1} \sum \{\varphi^z(n)\, Y_{\Psi_\eta}(zhz^{-1}) \colon z \in H\}.$$

Proof. As in the arguments above we have

$$Y_{\chi\Psi}(nh) = |H|^{-1}|H_\varphi|^{-1} \sum \{\varphi^z(n)(\varphi, \varphi^{shys^{-1}})_N \dot{\eta}^z(hy)\Psi(y^{-1}) \colon y, z \in H\},$$

and $(\varphi, \varphi^{shys^{-1}}) = 1$ if and only if $hy \in H_{\varphi^s}$. Thus

$$Y_{\chi\Psi}(nh) = |H|^{-1} \sum \Big\{\varphi^z(n) \cdot |H_\varphi|^{-1} \sum \{\dot{\eta}^z(hy)\Psi(y^{-1}) \colon hy \in H_{\varphi^z}\} \colon z \in H\Big\}.$$

Fix $z \in H$ and change variables: $v^{-1} = hy \in H_{\varphi^z}$. Then the inner summation becomes

$$|H_\varphi|^{-1} \sum \{\Psi(vh)\dot{\eta}^z(v^{-1}) \colon v \in H_{\varphi^z}\},$$

which is equal to

$$|H_\varphi|^{-1} \sum \{\Psi(hv)\eta^z(v^{-1}) \colon v \in H_{\varphi^z}\} = Y_{\Psi_{\eta^z}}(h),$$

since Ψ is a class function on H. An easy calculation shows that $Y_{\Psi_{\eta^z}}(h) = Y_{\Psi_\eta}(zhz^{-1})$, and so $(*)$ holds.

THEOREM 5.2. *H is multiplicity free in G if and only if H_φ is multiplicity free in H for every $\varphi \in \hat{N}$.*

Proof. $c_{\chi\Psi} = Y_{\chi\Psi}(1) = Y_{\Psi_\eta}(1) = c_{\Psi_\eta}$.

6. Dimensions of centralizer algebras. If G is any finite group and $K \leqslant G$, then the dimension (over C) of the centralizer algebra $(CG)^K$ is the number of K-orbits when K acts on G by conjugation.

THEOREM 6.1. *Suppose $N \triangleleft G$ and denote by M the number of G conjugacy classes in N. Then* $\dim(CG)^N = M \cdot [G:N]$.

Proof. By Burnside's orbit formula ([3], p. 68) we have

$$\dim(CG)^N = |N|^{-1} \sum \{|C_G(n)|:\ n \in N\} = |N|^{-1} \sum \{|C_N(x)|:\ x \in G\}$$
$$= [G:N]\,|G|^{-1} \sum \{|C_N(x)|:\ x \in G\} = [G:N]\cdot M,$$

since M is the number of G-orbits in N.

THEOREM 6.2. *If G is Frobenius with kernel N and complement H, then*

$$\dim(CG)^H = |N| + |\hat{H}| - 1.$$

Proof. Let $\varphi_1 = 1, \varphi_2, \ldots, \varphi_M$ be representatives of the H-orbits in $\hat{N}$ and let ϱ_H be the regular character of H. By the Frobenius orbit formula and the second orthogonality relation for characters we have

$$\dim(CG)^H = |H|^{-1} \sum \{|C_G(h)|:\ h \in H\}$$
$$= \sum \{(\varphi_i^G|H, \varphi_i^G|H):\ 2 \leqslant i \leqslant M\} + \sum \{(\chi|H, \chi|H):\ \chi \in \hat{G}, \chi|H \in \hat{H}\}$$
$$= \sum \{\varphi_i(1)^2(\varrho_H, \varrho_H):\ 2 \leqslant i \leqslant M\} + |\hat{H}|$$
$$= |H| \sum \{\varphi_i(1)^2:\ 2 \leqslant i \leqslant M\} + |\hat{H}|$$
$$= \sum \{\varphi(1)^2:\ \varphi \in \hat{N}, \varphi \neq 1_N\} + |\hat{H}| = |N| - 1 + |\hat{H}|,$$

since $|\mathrm{Orb}_H \varphi_i| = |H|$ if $\varphi_i \neq 1_N$.

THEOREM 6.3. *Suppose that G is a finite split extension NH, with N and H both abelian, and that M is the number of H-orbits in $\hat{N}$. Then*

$$\dim(CG)^H = M \cdot |H|.$$

Proof. If $\chi = \eta_\varphi^G \in \hat{G}$, then $\chi|H = [H:H_\varphi]\dot{\eta}$, so

$$(\chi|H, \chi|H)_H = [H:H_\varphi]^2 |H|^{-1} \sum \{\eta(h)\eta(h^{-1}):\ h \in H_\varphi\} = [H:H_\varphi].$$

Consequently,

$$\dim(CG)^H = \sum \{[H:H_{\varphi_i}]:\ \eta \in \hat{H}_{\varphi_i},\ 1 \leqslant i \leqslant M\}$$
$$= \sum \{[H:H_{\varphi_i}]\cdot|H_{\varphi_i}|:\ 1 \leqslant i \leqslant M\} = M \cdot |H|.$$

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