COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 2

TRANSFORMATIONS OF DIFFERENTIABLE FUNCTIONS

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Let $\mathcal{T}$ denote the set of real-valued functions defined on $[0,1]$ of the form $U \circ F(x)$, where $F(x)$ is a differentiable function and U is a homeomorphism. Equivalently, $\mathcal{T}$ consists of those continuous functions on $[0,1]$ which can be homeomorphically transformed into differentiable functions. Our principle result is that the condition (S′) defined in the sequel characterizes the class $\mathcal{T}$.

Since $F(x)$ is continuous, we assume that it maps $[0,1]$ into itself and that U is a homeomorphism of $[0,1]$ onto itself.

Definitions. 1. A continuous function F is said to be *ACG^* on an interval* I provided that

$$I = \bigcup_n E_n,$$

where each E_n is closed and, for any collection of intervals $I_k = [a_k, b_k]$ with end points in E_n, for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, n) > 0$ such that

$$\sum_k (b_k - a_k) < \delta \text{ implies } \sum_k |F(I_k)| < \varepsilon$$

($|E|$ denotes the Lebesgue measure of E).

2. A continuous function F is said to be *ACG on I* if

$$I = \bigcup_n E_n,$$

where each E_n is closed and, for any collection of intervals $I_k = [a_k, b_k]$ with end points in E_n, for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, n) > 0$ such that

$$\sum_k (b_k - a_k) < \delta \text{ implies } \sum_k |F(b_k) - F(a_k)| < \varepsilon.$$

3. A function F satisfies *Lusin's condition* (N) if $|E| = 0$ implies $|F(E)| = 0$.

4. A function F satisfies *Banach's condition* (T_1) if

$$|\{y \mid F^{-1}(y) \text{ is infinite}\}| = 0.$$

5. A function F satisfies *Banach's condition* (T_2) if

$$|\{y \mid F^{-1}(y) \text{ is uncountable}\}| = 0.$$

6. A function F satisfies *Banach's condition* (S) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|E| < \delta$ implies $|F(E)| < \varepsilon$.

7. A function F satisfies *condition* (S′) if for each open interval J contained in the range of F there exists $\varepsilon_J > 0$ such that $J \subset F(E)$ implies $|E| > \varepsilon_J$. (Only measurable sets E need be considered since a non-measurable set is always contained in a measurable set having the same outer measure.)

8. A function F satisfies *condition* (S″) if, for each open interval J contained in its range, $J \subset F(E)$ implies $|E| > 0$.

The following statements hold for continuous functions:

(i) (S) $\Rightarrow$ (S′) $\Rightarrow$ (S″).

(ii) (N) $\Rightarrow$ (S″).

(iii) ACG $\Rightarrow$ (N).

(iv) (T_1) and (N) $\Leftrightarrow$ (S).

(v) ACG* implies each of the properties in Definitions 2-8.

(vi) Differentiable functions satisfy ACG*.

(vii) (N) $\Rightarrow$ (T_2).

Statements (i) and (ii) follow easily from the definitions. Proofs for (iii)-(vii) can be found in [2], p. 277-289.

Before showing that $\mathcal{T}$ consists of the continuous functions which satisfy (S′), we note that (vi), (v), and (ii) imply that if F takes a set of measure zero onto a non-degenerate interval J, then $F \notin \mathcal{T}$; for if G is any homeomorphism, then $|G(J)| > 0$ and $G \circ F$ does not satisfy (N). Thus members of $\mathcal{T}$ must satisfy (S″), and Lebesgue's singular function (see [1], p. 113) provides an example of a continuous function which does not belong to $\mathcal{T}$, since it maps the Cantor set onto $[0, 1]$. Since every strictly increasing function belongs trivially to $\mathcal{T}$, the strictly increasing singular function given in [1], p. 278, shows that members of $\mathcal{T}$ need not satisfy (S).

THEOREM. *F belongs to $\mathcal{T}$ if and only if F is continuous and satisfies condition* (S′).

Proof. By (vi) and (v), every differentiable function satisfies conditions (T_1) and (N). Thus the necessity of (S′) can be established by assuming that F does not satisfy (S′) and by showing that if G is any homeomorphism, then $G \circ F$ cannot satisfy both (T_1) and (N).

If F does not satisfy (S′), then there are a non-degenerate interval $J \subset \operatorname{rng} F$ and a sequence of sets $\{E_n\}$ such that $|E_n| < 2^{-n}$ and $J \subset F(E_n)$ for all n. Let G be any homeomorphism. Then $G(J)$ is a non-degenerate interval and $G(J) \subset G\circ F(E_n)$ for all n.

Let

$$E = \overline{\lim_n} E_n = \{x \mid x \in E_k \text{ for infinitely many } k\} = \bigcap_n \bigcup_{k>n} E_k.$$

Since

$$\sum_n |E_n| < \infty,$$

we have $|E| = 0$. If $y \in G(J)$ is assumed by $G\circ F$ at only finitely many points $x_1, x_2, \ldots, x_s$, then one of these values must belong to infinitely many of the E_n. Hence $y \in G\circ F(E)$. Then, if $G\circ F$ satisfies (T_1),

$$|G\circ F(E) \cap G(J)| = |G(J)| > 0.$$

Since $|E| = 0$, $G\circ F$ does not satisfy (N) and the proof of the necessity is complete.

Suppose that F is continuous and satisfies (S′). We shall construct two homeomorphisms, G and H, of $[0, 1]$ onto itself such that $H\circ G\circ F$ is differentiable. Without loss of generality, assume that F maps $[0, 1]$ onto itself. Let (a, b) denote the open interval from a to b, where $a < b$ or $b < a$.

For $J = (y_1, y_2)$, set

$$\varepsilon(J) = \varepsilon(y_1, y_2) = \inf\{\varepsilon \mid \exists E \ni |E| = \varepsilon \text{ and } J \subset F(E)\}.$$

Set $G_0(y) = \varepsilon(0, y)$ for $y \in (0, 1]$ and set $G_0(0) = 0$.

(a) G_0 is strictly increasing.

Suppose that $0 < y_1 < y_2$ and $F(x_1) = y_1$. Given $\eta > 0$, choose E_0, E_1, and E_2 such that

$$E_0 \subset F^{-1}(0, y_2), \quad |E_0| < \varepsilon(0, y_2) + \eta \quad \text{and} \quad (0, y_2) \subset F(E_0);$$

$$E_1 \subset F^{-1}(0, y_1) \cap E_0, \quad |E_1| < \varepsilon(0, y_1) + \eta \quad \text{and} \quad (0, y_1) \subset F(E_1);$$

$$E_2 \subset F^{-1}(y_1, y_2) \cap E_0, \quad |E_2| < \varepsilon(y_1, y_2) + \eta \quad \text{and} \quad (y_1, y_2) \subset F(E_2).$$

Since the single point y_1 can be covered by $F(x_1)$, we have

$$\varepsilon(0, y_2) \leqslant |E_1| + |E_2| \leqslant \varepsilon(0, y_1) + \varepsilon(y_1, y_2) + 2\eta$$

and

$$\varepsilon(0, y_1) + \varepsilon(y_1, y_2) \leqslant |E_1| + |E_2| \leqslant \varepsilon(0, y_2) + \eta.$$

Since $\eta > 0$ is arbitrary, we obtain

$$\varepsilon(0, y_1) + \varepsilon(y_1, y_2) = \varepsilon(0, y_2).$$

Then $G_0(y_1)-G_0(y_2) = \varepsilon(y_1, y_2) > 0$ by condition (S′), and G_0 is strictly increasing.

(b) G_0 is continuous.

Let $\{y_n\}_{n=1}^{\infty}$ be an increasing sequence whose limit is y. Set $y_0 = 0$. Choose $\{x_n\}_{n=0}^{\infty}$ such that $F(x_n) = y_n$. Choose $E_n \subset F^{-1}(y_{n-1}, y_n)$ such that $(y_{n-1}, y_n) \subset F(E_n)$. Then the E_n are pairwise disjoint and

$$\sum_n |E_n| \leqslant 1.$$

Hence

$$\lim_{n\to\infty} \sum_{k\geqslant n} |E_k| = 0.$$

Since

$$(y_n, y) \subset F\Big(\bigcup_{k=n}^{\infty} E_k\Big) \cup F(\{x_n\}) \quad \text{and} \quad |\{x_n\}| = 0,$$

we have

$$G_0(y)-G_0(y_n) = \varepsilon(y_n, y) \leqslant \sum_{k\geqslant n} |E_k|$$

which tends to 0 as n tends to infinity. The proof for a decreasing sequence is analogous and the continuity of G_0 is established.

(c) For x_1, x_2 in $[0, 1]$,

$$\big|G_0\big(F(x_1)\big)-G_0\big(F(x_2)\big)\big| \leqslant |x_1-x_2|.$$

Since $\big(G_0\big(F(x_1)\big), G_0\big(F(x_2)\big)\big) \subset G_0(F[x_1, x_2])$, we have

$$\begin{aligned}|x_1-x_2| &\geqslant \inf\big\{|E| \,\big|\, \big(F(x_1), F(x_2)\big) \subset F(E)\big\}\\ &= \varepsilon\big(F(x_1), F(x_2)\big) = \big|G_0\big(F(x_1)\big)-G_0\big(F(x_2)\big)\big|.\end{aligned}$$

Noting that $G_0(1) > 0$, set $q = 1/G_0(1)$ and $G(y) = qG_0(y)$. It follows from (a) and (b) that G is a homeomorphism of $[0, 1]$ onto itself. Letting $K = G\circ F$, (c) implies that

$$|K(x_1)-K(x_2)| \leqslant q\,|x_1-x_2|$$

for all x_1, x_2 in $[0, 1]$.

This uniform Lipschitz condition implies that K is absolutely continuous. Consequently, the sets $Q = \{x \mid K'(x)$ does not exist$\}$ and $K(Q)$ are both of measure zero.

Let R be a G_δ-set of measure zero which contains $K(Q)$ and let $E = [0, 1]\setminus R$. It is clear from [3], Section 2, p. 3, and Theorem 8, p. 35, that E is a set for which there exists a derivative $h(t)$ such that $h(t) = 0$

for $t \in R$ and $0 < h(t) < 1$ for $t \in E$. Let

$$a = \int_0^1 h(t)\,dt$$

and set

$$H(y) = a^{-1} \int_0^y h(t)\,dt.$$

$H(y)$ is strictly increasing since $h(t) > 0$ on E and E is of full measure, $H(0) = 0$ and $H(1) = 1$. Since h is a derivative, $H'(y) = a^{-1}h(y)$ for all y. In particular, $H'(y) = 0$ for $y \in R$.

If $x_0 \notin Q$, then $K'(x_0)$ exists and $H \circ K$ is differentiable by the chain rule.

If $x_0 \in Q$, then $K(x_0) \in K(Q) \subset R$. For a point x at which $K(x) = K(x_0)$, we have

$$\left|H(K(x)) - H(K(x_0))\right| / |x - x_0| = 0.$$

If $K(x) \neq K(x_0)$, then

$$\left|H(K(x)) - H(K(x_0))\right| / |x - x_0| \leqslant q\left|H(K(x)) - H(K(x_0))\right| / |K(x) - K(x_0)|$$

by the Lipschitz condition. This last quantity tends to 0 as x tends to x_0, since $H'(y) = 0$ on R. Therefore, $H \circ K = H \circ G \circ F$ is differentiable at each x in $[0, 1]$ and the proof of the theorem is complete.

We note that ACG* functions belong to $\mathscr{T}$ since

$$\mathrm{ACG}^* \Rightarrow [(\mathrm{T}_1) \text{ and } (\mathrm{N})] \Leftrightarrow (\mathrm{S}) \Rightarrow (\mathrm{S}') \Leftrightarrow F \in \mathscr{T}.$$

We next show that there are ACG functions which do not belong to $\mathscr{T}$ and members of $\mathscr{T}$ which fail to satisfy (T_2).

Example 1. *There exists an ACG function F which does not belong to $\mathscr{T}$.*

Construction. The function F will be defined in such a way that $F^{-1}(y)$ is infinite for all but a countable number of values y in its range. Then, if G is any homeomorphism, $(G \circ F)^{-1}(z)$ is infinite for all but a countable number of values z in its range. Since $G \circ F$ does not satisfy (T_1), it cannot be differentiable and $F \notin \mathscr{T}$.

Let C be a Cantor set of positive measure such that, for open intervals I,

$$I \cap C \neq \emptyset \Rightarrow |I \cap C| > 0.$$

Let $f(x) = 0$ if $x \notin C$, $f(x) = 1$ if $x \in C$ and

$$F_0(x) = \int_0^x f(t)\,dt.$$

For each integer n, select a sequence of intervals $I_n = (a_n, b_n)$ contiguous to C so that

$$\lim_{n\to\infty} b_n = 1, \quad \lim_{n\to-\infty} a_n = 0 \quad \text{and} \quad n < n' \Rightarrow a_n < a_{n'}.$$

Let

$$c_n = \tfrac{1}{2}(a_n + b_n)$$

and define $F_1(x)$ to be $F_0(x)$ if $x \notin \bigcup I_n$, $F_1(c_n) = F_0(b_{n-1})$ and define $F_1(x)$ linearly on the intervals (a_n, c_n) and (c_n, b_n). Assuming that $F_k(x)$ has been defined, define $F_{k+1}(x)$ inductively as follows. For each integer n, select intervals

$$I_{n_1 n_2 \ldots n_k n} = (a_{n_1 n_2 \ldots n_k n}, b_{n_1 n_2 \ldots n_k n})$$

contiguous to C so that

$$\lim_{n\to\infty} b_{n_1 n_2 \ldots n_k n} = a_{n_1 n_2 \ldots n_k + 1}, \quad \lim_{n\to-\infty} a_{n_1 n_2 \ldots n_k n} = b_{n_1 n_2 \ldots n_k}$$

and

$$n < n' \Rightarrow a_{n_1 n_2 \ldots n_k n} < a_{n_1 n_2 \ldots n_k n'}.$$

Let

$$c_{n_1 n_2 \ldots n_k n_{k+1}} = \tfrac{1}{2}(a_{n_1 n_2 \ldots n_k n_{k+1}} + b_{n_1 n_2 \ldots n_k n_{k+1}})$$

and put

$$F_{k+1}(x) = F_k(x) \quad \text{if } x \notin \bigcup I_{n_1 n_2 \ldots n_k n_{k+1}}$$

(the union extends over all $(k+1)$-tuples),

$$F_{k+1}(c_{n_1 n_2 \ldots n_k n_{k+1}}) = F_k(b_{n_1 n_2 \ldots n_k n_{k+1} - 1})$$

and define F_{k+1} linearly on the intervals

$$(a_{n_1 n_2 \ldots n_k n_{k+1}}, c_{n_1 n_2 \ldots n_k n_{k+1}}) \quad \text{and} \quad (c_{n_1 n_2 \ldots n_k n_{k+1}}, b_{n_1 n_2 \ldots n_k n_{k+1}}).$$

Since $F_k(x)$ is a uniformly convergent sequence of continuous functions,

$$F(x) = \lim_k F_k(x)$$

is continuous. We observe that $F(x)$ is ACG by noting that $F(x)$ is absolutely continuous on C and on each interval contiguous to C. It follows from the construction that, given any $y \in \operatorname{rng} F_0$ with $y \neq F_0(1)$, $y \neq F_0(0)$, and $y \neq F_0(t)$ for t an end point of an interval contiguous to C, and for each natural k, y belongs to some interval $\big(F_0(b_{n_1 n_2 \ldots n_k}), F_0(a_{n_1 n_2 \ldots n_k + 1})\big)$. Thus, for each k, there is an $x_k \in \bigcup I_{n_1 n_2 \ldots n_{k+1}}$ such that

$$x_k \in (b_{n_1 n_2 \ldots n_k}, a_{n_1 n_2 \ldots n_k + 1}) \quad \text{and} \quad F_{k+1}(x_k) = F(x_k) = y.$$

It follows that $F^{-1}(y)$ is infinite except on an at most countable collection of $y \in \operatorname{rng} F_0 = \operatorname{rng} F$.

We note that this F satisfies (N) and hence (S″), which indicates that (S′) and (S″) are not equivalent.

Example 2. *There is a function F which belongs to $\mathscr{T}$ but does not satisfy* (T_2).

Construction. For each number x in the Cantor ternary set C write

$$G(x) = \sum_i a_{2i}\cdot 3^{-i}, \quad \text{where } x = \sum_i a_i \cdot 3^{-i},\ a_i = 0 \text{ or } 2.$$

Extend $G(x)$ to a continuous function on $[0, 1]$ by defining $G(x)$ to be linear on intervals contiguous to C. If

$$y \in C \quad \text{and} \quad y = \sum_i b_i \cdot 3^{-i},$$

then each number x of the form

$$\sum_i a_i \cdot 3^{-i}, \quad \text{where } a_{2i} = b_i \text{ and } a_{2i-1} = 0 \text{ or } 2,$$

maps onto y. Therefore, G maps C onto C and, for all $y \in C$, $G^{-1}(y)$ is uncountable. Since G takes C onto C and is linear on intervals contiguous to C, G satisfies (N) and hence (T_2). In fact, if y_0 belongs to an interval J contiguous to C, the line $y = y_0$ meets only finitely many of the linear segments of the graph of $G(x)$. (All of these linear segments are of length greater than $|J|$ and, due to the continuity of G, there can only be finitely many meeting $y = y_0$.) Now $G(x)$ also belongs to $\mathscr{T}$. For let J be an interval in the range of G, let $J_0 \subset J$ be an interval contiguous to C, and let M be the maximum of the absolute values of the slopes of the finitely many linear segments which cross a given line $y = y_0$ with $y_0 \in J_0$. Then, if G takes a set E onto J, then since $|G'(x)| \leqslant M$ on $E' = E \cap G^{-1}(J_0)$, we have $|J_0| = |G(E')| \leqslant M|E'|$ so that $|E| \geqslant M^{-1}|J_0|$. Since $M > 0$, $M^{-1}|J| > 0$ and G satisfies (S′) and hence belongs to $\mathscr{T}$. Now let $K(y)$ be a homeomorphism which takes C onto a perfect set C' of positive measure. Let $F = K \circ G$. Then, for each $y \in C'$, $F^{-1}(y)$ is uncountable so that F does not satisfy (T_2). However, $F = K \circ G$, $G \in \mathscr{T}$, K is a homeomorphism, and thus $F \in \mathscr{T}$.

The following proposition and example show the extent to which members of $\mathscr{T}$ must be differentiable.

PROPOSITION. *If F belongs to $\mathscr{T}$, then $\{x \mid F'(x) \text{ exists}\}$ has positive measure in each subinterval of* $[0, 1]$.

Proof. Let $F = H \circ D$, where D is differentiable and H is a homeomorphism. Let J be a subinterval of $[0, 1]$. If D is constant on J, then

$F'(x) = 0$ on J. Otherwise, $D(J)$ is a non-degenerate interval and, since H is a homeomorphism, the set

$$Q = \{y \in D(J) \mid H'(y) \text{ exists}\}$$

satisfies $|Q| = |D(J)| > 0$. If $x \in D^{-1}(Q) \cap J$, then $F'(x)$ exists by the chain rule. In view of $D(D^{-1}(Q) \cap J) = Q$ we have $|D^{-1}(Q) \cap J| > 0$, since D satisfies condition (N).

Example 3. *There exists a function F in $\mathscr{T}$ which fails to be differentiable on a set of positive measure.*

Construction. An ACG function F which fails to be differentiable on a set of positive measure is constructed in [2], p. 224. Since F is ACG, it satisfies condition (N) and it is easy to verify that it satisfies condition (T_1). Thus F belongs to $\mathscr{T}$.

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Reçu par la Rédaction le 10. 8. 1976