COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 1

OPEN, CONFLUENT AND RELATED MAPPINGS ON GENERALIZED GRAPHS

BY

S. MIKLOS (WROCŁAW)

In 1976 C. A. Eberhart, J. B. Fugate, and G. R. Gordh, Jr. (see [3], p. 413) asked "Is every graph the weakly confluent image of a finite tree?". In the same year, using universal covering space techniques, B. B. Epps published an affirmative answer to this question (see [5], Theorem 1, p. 220). Moreover, it is easy to show that no cyclic graph is the image of a finite tree under locally confluent (thus also under confluent and open, respectively) mapping. In the paper [2] a concept of generalized graph is introduced. In this paper it is shown that 1) each generalized graph is the open image of a generalized tree, 2) each generalized graph is the weakly confluent image of a generalized tree, and 3) a cyclic generalized graph is a simple closed curve if and only if it is the locally confluent image of a generalized tree. As a consequence of 2) we obtain Epps' Theorem 1 mentioned above.

The proofs, presented below, are quite elementary and use geometrical methods.

All spaces considered here are metric. A *graph* means a one-dimensional connected polytope. A space is called a *generalized graph* if it is connected and if it can be embedded into a graph. An acyclic generalized graph is called a *generalized tree*. We denote by $R(G)$ the set of all ramification points and by $E(G)$ the set of all end points of the generalized graph G, and we put $V(G) = R(G) \cup E(G)$. An arc ab contained in the graph G is called a *maximal free arc of* G provided that $\{a, b\} = ab \cap V(G)$. A continuous surjection $f\colon X \to Y$ of a topological space X onto a topological space Y is said to be

– *open* if the image of any open set in X is open in Y;

– *confluent* if for every continuum $Q \subset Y$ and for each component C of $f^{-1}(Q)$ we have $f(C) = Q$;

– *weakly confluent* if for every continuum $Q \subset Y$ there is a component C of $f^{-1}(Q)$ such that $f(C) = Q$.

THEOREM 1. *Each generalized graph is the open image of a generalized tree.*

Proof. Let a generalized graph Y be given. Since a circle S is the image of a generalized arc under an open mapping g: $(0, 1) \to S$ given by $g(t) = \exp(3\pi i t)$, we assume that Y is not a circle. Moreover, if Y is a generalized tree, then the theorem is trivial, so without loss of generality we can assume that Y is not a generalized tree, too. Let

$$(1) \qquad A_1, A_2, \ldots, A_n$$

be a sequence of all maximal free arcs and of all simple closed curves with exactly one ramification point that are contained in Y.

We construct a generalized tree T contained in Y as follows. According to the assumption, Y contains a simple closed curve. We take an arbitrary member A_i of (1) lying in a simple closed curve in Y, and we denote by B_1 an arc contained in Int A_i. Put $Y_1 = Y \setminus B_1$. If Y_1 is a generalized tree, we define $T = Y_1$. If not, a simple closed curve is in Y_1, and the inductive procedure is continued. Since (1) is a finite sequence, the inductive procedure is finite and it leads to a generalized tree T. Therefore, there is a natural number $k \leqslant n$ such that

$$(2) \qquad T = Y_k \setminus B_k = Y \setminus \bigcup_{j=1}^{k} B_j.$$

To define a continuous surjection f: $T \to Y$ which is open we need some auxiliary denotations. In each member A_i of (1) that contains some B_j, where $j \in \{1, 2, \ldots, k\}$ (see (2)), we take a maximal free arc wz contained in the closure of T with $w \in R(T)$ and $z \notin T$. Let U denote the union of all such arcs wz. Put $W = \overline{T \setminus U}$.

Now we define f as follows. $f|W$ is the identity, while $f|wz \setminus \{z\}$ is any homeomorphism of $wz \setminus \{z\}$ into A_i with $f(w) = w$ and $f(wz \setminus \{z\}) \supset B_j$, and such that both x and $f(x)$ are contained in $wz \setminus \{z\}$ whenever x lies closely enough to w.

It is easy to verify that f is well defined, continuous, open and onto. The proof is complete.

Now, we recall the second part of a well-known result of Whyburn (see Theorem (1.1) in [8], p. 182) which says that if f: $X \to Y$ is an open mapping from a linear graph X onto a metric space Y, and if J is any simple closed curve in Y, then there exists a simple closed curve C in X such that $f(C) = J$ and, on C, f is topologically equivalent to the function $w = z^k$ on the unit circle $|z| = 1$, where k is an integer.

This result and Theorem 1 imply

Corollary 1. *Each cyclic generalized graph is the open image of a generalized tree but never of a tree.*

Now, we come back to the function f: $T \to Y$ defined in the proof of

Theorem 1, and note that

(3) f is not weakly confluent (thus it is not confluent).

Indeed, let B_j ($j \in \{1, 2, \ldots, k\}$) be any arc defined above. Then there is a member A_i of (1) such that $B_j \subset A_i$. By the second part of the definition of f we see that $f^{-1}(A_i)$ consists of two different generalized arcs, both denoted by $wz \setminus \{z\}$, but no one of them is mapped onto A_i. So (3) is established.

However we have

THEOREM 2. *Each generalized graph is the weakly confluent perfect image of a generalized tree.*

Proof. Let a generalized graph Y be given. Since a circle S is the image of an arc under a weakly confluent mapping g: $[0, 1] \to S$ given by $g(t) = \exp(4\pi i t)$, we assume that Y is not a circle. Moreover, we assume (as in the proof of Theorem 1) that Y is not a generalized tree. Let sequence (1) have the same meaning as previously in the proof of Theorem 1. Put $A = \bigcup_{i=1}^{n} A_i$, and note that $A = Y$ if Y is compact. Otherwise $Y \setminus A$ is the union of some generalized arcs (arcs without their end points). Denote by $T(Y)$ a family of all generalized trees T contained in Y such that each component of $Y \setminus T$ is an arc without its end points, $Y \setminus A \subset T$ and $V(Y) \cap \overline{Y \setminus T} = \emptyset$.

Note that

(4) for each subgraph Q of Y there is a generalized tree $T \in T(Y)$ such that $T \cap Q$ is connected and, moreover, it is a member of the family $T(Q)$, where $T(Q)$ is defined analogously to $T(Y)$ (i.e., if $T' \in T(Q)$ then $V(Q) \cap \cap \overline{Q \setminus T'} = \emptyset$).

Indeed, Y contains a simple closed curve. If there exists a member A_i of (1) lying in simple closed curve in Y and not contained in Q, let B_{i_1} denote an open arc such that $B_{i_1} \subset \bar{B}_{i_1} \subset \operatorname{Int} A_{i_1} \setminus Q$. If there is no such A_{i_1} we let A_{i_1} be any of them (contained in simple closed curve in Y) and we denote by B_{i_1} an open arc such that $B_{i_1} \subset \bar{B}_{i_1} \subset A_{i_1}$. Put $Y_1 = Y \setminus B_{i_1}$. If Y_1 is a generalized tree, we define $T = Y_1$. If not, a simple closed curve is in Y_1, and inductive procedure is continued.

It follows from the construction that $T \cap Q$ is connected. Thus $T \cap Q$ is a tree. By definition of $T(Q)$ we see that $T \cap Q \in T(Q)$. Since (1) is a finite sequence, the procedure is finite. So (4) is established.

Now we define an equivalence relation ϱ on $T(Y)$ putting $(T_1, T_2) \in \varrho$ provided that for every A_i of (1) we have $A_i \subset T_1$ if and only if $A_i \subset T_2$. Consider the quotient space $T(Y)/\varrho$, and note that

(5) the set $T(Y)/\varrho$ (whose elements are the ϱ-equivalence classes) is finite.

We put $m = \operatorname{card}(T(Y)/\varrho)$ and let $\{T_1, T_2, \ldots, T_m\}$ be any system of representatives for $T(Y)/\varrho$, i.e., a set of elements, one from each equivalence class. Further fix a point $r \in V(Y)$. Since $V(Y) \subset T_j$ for each $j \in \{1, 2, \ldots, m\}$, we have $r \in \bigcap \{T_j\colon j \in \{1, 2, \ldots, m\}\}$. Consider the free union $\bigoplus_{j=1}^{m} T_j$ of all generalized trees T_j and let r_j be the copy of the point r in T_j for each $j \in \{1, 2, \ldots, m\}$. Thus $r_j \in T_j$.

Further, let σ denote an equivalence relation on $\bigoplus_{j=1}^{m} T_j$ which identifies the points r_j for every $j \in \{1, 2, \ldots, m\}$ only, i.e., all equivalence classes of σ except $\{r_1, r_2, \ldots, r_m\}$ are one point sets. Then the quotient space $\bigoplus_{j=1}^{m} T_j/\sigma$ is a generalized tree consisting of copies of the generalized trees T_j ($j \in \{1, 2, \ldots, m\}$), i.e., it is homeomorphic to the one-point union of the generalized trees T_j. We put $\bigoplus_{j=1}^{m} T_j/\sigma = X$. So X is a generalized tree.

To define a continuous surjection $f\colon X \to Y$ which is weakly confluent we need some auxiliary denotations.

Let $p\colon \bigoplus_{j=1}^{m} T_j \to X$ be a natural projection of $\bigoplus_{j=1}^{m} T_j$ onto X. Furthermore, for every $j \in \{1, 2, \ldots, m\}$ let $g_j\colon T_j \to Y$ be a natural embedding of T_j into Y and let $q\colon \bigoplus_{j=1}^{m} T_j \to Y$ be their unique common extension (i.e., $q|T_j = q_j$ for each $j \in \{1, 2, \ldots, m\}$). We put $V = p(q^{-1}(V(Y)))$ and let W be a generalized subtree of X such that $E(W) \subset V \subset W$ and $W \supset p(q^{-1}(Y \setminus A))$ ($W \neq \emptyset$ because Y is not a circle).

Note that the closure of each component of $X \setminus W$ is an arc with one end point in W. Take an arbitrary arc wz being the closure of a component of $X \setminus W$, where $w \in W$, and put $w'z' = q(p^{-1}(wz)) \subset Y$, where $q(p^{-1}(w)) = w' \in V(Y)$. Then there is exactly one A_i of (1) such that $w'z' \subset A_i$.

Now we define f as follows (see the diagram):

(i) $f|W\colon W \to Y$ is defined by $f(w) = q(p^{-1}(w))$ for each point $w \in W$.

(ii) If a component of $X \setminus W$ is given, we take the arc wz being its closure, with $w \in W$, and a member A_i of (1) such that $w'z' \subset A_i$. Consider two cases. First, if A_i is an arc, then we define $f|wz$ as an arbitrary homeomorphism of wz onto A_i with $f(w) = w$. Second, if A_i is a simple closed curve, then we stretch out the arc wz onto A_i such that for each point $x \in wz$ which lies closely enough to w, its image $f(x)$ is in $w'z'$.

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& \bigoplus_{j=1}^{m} T_j \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

It is easy to verify that f is well defined, continuous and onto.

We shall show that f is weakly confluent. To this end we take any subcontinuum Q in Y. Assume first $Q \cap V(Y) = \emptyset$. Then Q is the arc contained in some A_i of (1). Therefore, for each $j \in \{1, 2, \ldots, m\}$ there is an arc $E_j \subset T_j$ such that the partial mapping $f \mid p(E_j)$ is a homeomorphism. Hence $p(E_j)$ is the component of $f^{-1}(Q)$ and $f(p(E_j)) = Q$. Assume second $Q \cap V(Y) \neq \emptyset$. Then by (4) there is $T \in T(Y)$ such that $T \cap Q \in T(Q)$. Let $j \in \{1, 2, \ldots, m\}$ be an index such that $(T, T_j) \in \varrho$. Thus $q_j(T_j) \cap Q$ is a member of the family $T(Q)$. We put $P' = q_j(T_j) \cap Q$ and $P = p(q_j^{-1}(P'))$. We see that P is homeomorphic to P'. Further, since $f \mid P \cap W$ is a homeomorphism (see (i)) and the arc $wz \subset \overline{P \setminus W}$ is mapped onto the whole $A_i \supset w'z'$ (see (ii)), we conclude that $f(P)$ contains Q. If $f(P) = Q$, then $f^{-1}(Q)$ containing P is mapped onto Q under f. If $f(P) \neq Q$ we consider any maximal free arc $wz \subset P$ with $f(wz) \setminus Q \neq \emptyset$. By (ii) there is a subarc $we \subset wz$ such that $f(we) = w'e'$, where $e' \subset Q \cap \overline{f(P) \setminus Q}$ and moreover $f \mid we$ is a homeomorphism. Further, we have $f(we) \subset Q$. Let Z denote the union of all arcs $ez \subset wz$ with $f(wz) \setminus Q \neq \emptyset$, and put $D = \overline{P \setminus Z}$. The set D is connected because Z is the union of end-arcs of P. So D is a continuum. It follows from $f(P) \supset Q$ and from the definition of D that $f(D) = Q$. Thus $D \subset f^{-1}(Q)$. Hence there is a component C of $f^{-1}(Q)$ containing D. Therefore $Q = f(D) \subset f(C) \subset Q$, so $f(C) = Q$.

Finally we show that f is a perfect mapping. Indeed, for each point y in Y we have $\operatorname{card} f^{-1}(y) \leqslant 2m < \aleph_0$. To observe that f is closed we note that if T_j' in X is the copy of T_j in Y then $f \mid T_j'$ is a closed mapping, for each $j \in \{1, 2, \ldots, m\}$. The proof is complete.

As a consequence of Theorem 2 we get

COROLLARY 2. (Epps' Theorem). *Each graph is the weakly confluent image of a finite tree.*

In fact, compactness is an inverse invariant of perfect mappings (see Theorem 3.7.24 of [4], p. 242).

Remark 1. A well-known result of Whyburn (see Theorem (7.5) in [8], p. 148) says that if f is an open mapping from a compact set, then f is confluent. By (3) above we see that the compactness assumed there is essential to attain the conclusion.

Remark 2. A simple closed curve is the confluent image of a generalized tree.

Indeed, we take a unit circle $S = \{z: |z| = 1\}$, where z is a complex number, and let I denote the interval $(-\pi/2, \pi/2)$ of the real line $\boldsymbol{R}$. Further, we define two continuous surjections: $f\colon I \to \boldsymbol{R}$, $x \to \operatorname{tg} x$; $g\colon \boldsymbol{R} \to S$, $x \to (\cos 2\pi x, \sin 2\pi x)$. Put $h = gf$. It is easy to verify that h is confluent and closed (cf. [7], Theorem 3.2, p. 186).

THEOREM 3. *No cyclic generalized graph except the simple closed curve is the locally confluent image of a generalized tree.*

Proof. Suppose the contrary, and let Y denote any cyclic generalized graph which is different from a simple closed curve and such that there is a generalized tree T and a locally confluent mapping f from T onto Y. Since local confluency of a mapping onto a hereditarily arcwise connected space is equivalent to its confluency (see 5.3 of [6], p. 110) we conclude that f is confluent. Further, let S be any simple closed curve in Y, and C a component of $f^{-1}(S)$. Thus $f|C\colon C \to S$ is confluent and C is not a tree because the confluent image of a tree is a tree (see [1], X, p. 216). Thus there is a tree G such that $G \setminus C$ is a finite set (see Theorem 1 (iii) of [2], p. 337). We put $E(G) \setminus E(C) = \{e_1, e_2, \ldots, e_n\}$ and let $a_1 e_1, a_2 e_2, \ldots, a_n e_n$ be arcs in G such that for every $i \in \{1, 2, \ldots, n\}$, $a_i \notin R(T)$ and the arc $a_i e_i \setminus \{a_i, e_i\}$ is open in T. We note that there is an index $j \in \{1, 2, \ldots, n\}$ such that $f(a_j e_j \setminus \{e_j\})$ is non-degenerate. By assumption S contains at least one ramification point of Y. Therefore, by confluency of f we infer that $\overline{f(a_j e_j \setminus \{e_j\})}$ is an arc in S with end points $f(a_j)$ and e. Now, let U be a closed connected neighborhood of e such that $(U \setminus \{e\}) \cap (V(Y) \cup \{f(a_j)\}) = \emptyset$, and let K be a component of $f^{-1}(U)$ such that $(a_j e_j \setminus \{e_j\}) \cap K \neq \emptyset$. Thus we have $U \setminus f(K) \neq \emptyset$ contrary to confluency of f. The proof is complete.

Let us note, in connection with Theorem 3 and Remark 2, that we have the following characterization of a simple closed curve.

COROLLARY 3. *A cyclic generalized graph is a simple closed curve if and only if it is the locally confluent image of a generalized tree.*

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
WROCŁAW, POLAND.

Reçu par la Rédaction le 20. 10. 1982,
en version modifiée le 03. 08. 1983