COLLOQUIUM MATHEMATICUM

VOL. XLIII 1980 FASC. 2

SOME REMARKS ON THE INTERFACE IN THE FILTRATION PROBLEM

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We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2}, \tag{1}$$

$$u(x, 0) = u_0(x), \tag{2}$$

where $\varphi(u)$ is a given function defined for $u \geqslant 0$, and $u_0(x) \geqslant 0$ is a bounded continuous function in R.

Definition. Let $S = \{(x, t): x \in R, t > 0\}$. A function $u(x, t) \geqslant 0$ defined in S is a *weak solution* of problem (1), (2) if

$$u \in C^0(\bar{S}) \cap L^\infty(S), \qquad \frac{\partial \varphi\big(u(x, t)\big)}{\partial x} \in L^\infty(S), \qquad u(\cdot, 0) = u_0$$

and

$$\iint_S \left[\frac{\partial \Phi}{\partial x}\frac{\partial \varphi(u)}{\partial x} - \frac{\partial \Phi}{\partial t} u\right] dx\, dt = \int_{-\infty}^{\infty} \Phi(x, 0) u_0(x)\, dx \tag{3}$$

for all $\Phi \in C_0^1(\bar{S})$.

Hereafter we assume that

(4) $\varphi(u_0)$ is Lipschitz continuous on R;

(5) $\varphi \in C^5\big((0, +\infty)\big)$, $\varphi^{(5)}$ is Lipschitz continuous on any compact subset of $(0, +\infty)$;

(6) $\varphi(u) > 0$, $\varphi'(u) > 0$ for $u > 0$, and $\varphi(0) = \varphi'(0) = 0$;

(7) $\displaystyle\int_0^u \frac{\varphi'(a)}{a}\, da < \infty$ for all u;

(8) for every $N > 0$ there exist constants a, β, γ ($a > 0, 0 < \beta < 1, \gamma > 0$) such that $\varphi''(u)/(\varphi'(u))^2 \geqslant a$ and $\varphi'(u) \geqslant \gamma\varphi^\beta(u)$ for $0 < u \leqslant N$.

Oleĭnik et al. [5] have shown that if conditions (4)-(6) hold, then problem (1), (2) has a unique weak solution u, and u satisfies equation (1) in the classical sense in the neighbourhood of every point of S at which u is positive.

Let $\operatorname{supp} u_0 = [a_1, a_2]$ and let u be a weak solution of problem (1), (2). Then from Theorem 21 and Remark 2 in [5] it follows that the set $P[u] = \{(x, t) \in S\colon u(x, t) > 0\}$ is bounded by the interval $[a_1, a_2]$, by a continuous monotone non-increasing curve $x = \zeta_1(t)$ through $(a_1, 0)$, and by a continuous monotone non-decreasing curve $x = \zeta_2(t)$ through $(a_2, 0)$.

Aronson [3] has studied the case in which $\varphi(u) = u^m$ ($m > 1$) and has obtained a more precise characterization of the interface, that is, of curves $x = \zeta_i(t)$.

From our assumptions imposed upon φ and from the implicit function theorem it follows that the equation

$$\int\limits_0^u \frac{\varphi'(a)}{a}\,da = v$$

defines

$$u = \psi(v) \tag{9}$$

in a unique way. Putting (9) into (1) and (2) we get

$$\frac{\partial v}{\partial t} = \varphi'\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 \tag{10}$$

and

$$v(x, 0) = v_0(x). \tag{11}$$

A function $v(x, t)$ is said to be a *weak solution* of problem (10), (11) in S if $u = \psi(v)$ is a weak solution of problem (1), (2) in S with initial data $u_0 = \psi(v_0)$.

THEOREM. *Let $u_0 \in C^0(R) \cap L^\infty(R)$, $u_0(x) \geqslant 0$, $\operatorname{supp} u_0 = [a_1, a_2]$ and let conditions* (4)-(8) *be satisfied. Assume that*

$$\left|\frac{\partial^2 v(x, t)}{\partial x^2}\right| \leqslant M \quad \textit{for all } t \in [\tau, T],\ x \in \big(\zeta_1(t), \zeta_2(t)\big) \ (0 < \tau < T < \infty),$$

where $v(x, t)$ is given by (9) *and $u(x, t)$ is a weak solution of problem* (1), (2).

Then $\zeta_i \in C^1([\tau, T])$, and the limits

$$\lim_{x \to \zeta_i(t)} \frac{\partial v(x, t)}{\partial x} \equiv \frac{\partial}{\partial x} v\big(\zeta_i(t), t\big) \quad (i = 1, 2)$$

exist for all $t \in [\tau, T]$ and

$$\zeta_i'(t) = -\frac{\partial}{\partial x} v(\zeta_i(t), t) \quad \textit{for all } t \in [\tau, T]$$

(here and elsewhere in this paper $x \to \zeta_i(t)$ means $x \to \zeta_i(t) - (-1)^i 0$).

Proof. We observe that the boundedness of the function $\partial^2 v/\partial x^2$ for $t \in [\tau, T]$ and $x \in (\zeta_1(t), \zeta_2(t))$ implies that the limits

$$\lim_{x \to \zeta_i(t)} \frac{\partial v(x, t)}{\partial x} = \frac{\partial}{\partial x} v(\zeta_i(t), t) \quad (i = 1, 2)$$

exist and are finite.

We now prove the continuity of the functions

$$\frac{\partial}{\partial x} v(\zeta_i(t), t) \quad \text{in } [\tau, T].$$

Let t_n, $t \in [\tau, T]$, $t_n \to t$. Define the functions f_n^i and f^i by

$$f_n^i(x) = v(x + \zeta_i(t_n) - a_i, t_n), \quad f^i(x) = v(x + \zeta_i(t) - a_i, t),$$

where $i = 1, 2$, $n = 1, 2, \ldots$, $x \in [a_1, a_2]$. We can prove by the arguments similar to those used in the proof of the Theorem in [1] that there exists a constant $C > 0$ such that

$$\left| \frac{\partial v}{\partial x} \right| \leqslant C \quad \text{for } t \in [\tau, T],\ x \in (\zeta_1(t), \zeta_2(t)). \tag{12}$$

If conditions (4)-(6) and (8) are fulfilled, then by Theorem 2 from [6] there exists a continuous non-decreasing function $\varrho(s)$ ($\varrho(0) = 0$) such that

$$|u(x, t) - u(x, s)| \leqslant \varrho(|t - s|) \quad \text{for } x \in R,\ t \geqslant \tau,\ s \geqslant \tau\ (\tau > 0). \tag{13}$$

It is easy to verify that the inequality

$$v_2 - v_1 \leqslant \psi^{-1}(\psi(v_2) - \psi(v_1)) + 2 \sup_{u \leqslant \psi(v_2)} \varphi'(u) \frac{\psi(v_2) - \psi(v_1)}{\psi(v_2)} \tag{14}$$

holds, where $0 \leqslant v_1 < v_2$, and the function ψ is defined by

$$\psi^{-1}(u) = \int_0^u \frac{\varphi'(a)}{a} da.$$

Hence, by (12)-(14), the function $v(x, t)$ is uniformly continuous on the set $\{(x, t)\colon x \in (\zeta_1(t), \zeta_2(t)),\ t \in [\tau, T]\}$. This implies

$$f_n^i \underset{n}{\rightrightarrows} f^i.$$

The uniform boundedness of the functions $\{(f_n^i)''\}$ implies that the sequences $\{(f_n^i)'\}$ $(i = 1, 2)$ are compact in uniform convergence topology. Therefore (since $f_n^i \underset{n}{\rightrightarrows} f^i$)

$$(f_n^i)' \underset{n}{\rightrightarrows} (f^i)' \quad \text{and} \quad \lim_{n\to\infty} \frac{\partial}{\partial x} v(\zeta_i(t_n), t_n) = \frac{\partial}{\partial x} v(\zeta_i(t), t).$$

The Theorem follows immediately from

LEMMA. *If the assumptions of the Theorem hold true, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left| \frac{\zeta_i(t) - \zeta_i(s)}{t-s} + \frac{\partial}{\partial x} v(\zeta_i(s), s) \right| < \varepsilon \quad (i = 1, 2) \tag{15}$$

for $s, t \in [\tau, T]$, $0 < t-s < \delta$.

Proof. We prove inequality (15) for ζ_1.

Let $\tau \leqslant s < T$ and $\varepsilon > 0$. For convenience we write

$$\zeta_1 = \zeta_1(s) \quad \text{and} \quad p = \frac{\partial}{\partial x} v(\zeta_1, s).$$

By repeated integration, with respect to x in the interval (ζ_1, x) $(x \in (\zeta_1(s), \zeta_2(s)))$, of the inequalities

$$-M \leqslant \frac{\partial^2}{\partial x^2} v(x, s) \leqslant M$$

we obtain

$$-\frac{M}{2}(x-\zeta_1)^2 \leqslant v(x, s) - p(x-\zeta_1) \leqslant \frac{M}{2}(x-\zeta_1)^2.$$

Hence

$$-\frac{M}{2}(x-\zeta_1)^2 + p(x-\zeta_1) \leqslant v(x, s) \leqslant \frac{M}{2}(x-\zeta_1)^2 + p(x-\zeta_1). \tag{16}$$

It is easy to verify that the functions

$$v_1(x, t) = \max\{0,\ (p-\varepsilon)(x-\zeta_1) + (p-\varepsilon)^2(t-s)\},$$
$$v_2(x, t) = \max\{0,\ (p+\varepsilon)(x-\zeta_1) + (p+\varepsilon)^2(t-s)\}$$

are weak solutions of (10) in the domain $R \times [s, T]$.

From (16) it follows that

$$v(x,s)-v_1(x,s)\geqslant \varepsilon(x-\zeta_1)-\frac{M}{2}(x-\zeta_1)^2,$$

$$v_2(x,s)-v(x,s)\geqslant \varepsilon(x-\zeta_1)-\frac{M}{2}(x-\zeta_1)^2,$$

whence

$$v_1(x,s)\leqslant v(x,s)\leqslant v_2(x,s)\qquad \text{for } x\leqslant x_0=\zeta_1+\frac{\varepsilon}{M},$$

$$v(x_0,s)-v_1(x_0,s)\geqslant \frac{\varepsilon^2}{2M},\qquad v_2(x_0,s)-v(x_0,s)\geqslant \frac{\varepsilon^2}{2M}.$$

The last inequalities imply

$$u_1(x,s)\leqslant u(x,s)\leqslant u_2(x,s)\qquad \text{for } x\leqslant x_0, \tag{17}$$

$$u(x_0,s)-u_1(x_0,s)\geqslant C,\qquad u_2(x_0,s)-u(x_0,s)\geqslant C, \tag{18}$$

where $u_i=\psi(v_i)$ $(i=1,2)$, $u=\psi(v)$, the function ψ is defined by

$$\psi^{-1}(v)=\int_0^u \frac{\varphi'(a)}{a}\,da,$$

and the constant $C>0$ is independent of s. Inequalities (18) follow at once from (14). By (13) and (18) we get

$$u_1(x_0,t)\leqslant u(x_0,t)\leqslant u_2(x_0,t)\qquad \text{for } s\leqslant t\leqslant s+\delta,\ \delta=\varrho^{-1}\left(\frac{C}{2}\right). \tag{19}$$

From (17), (19) and from Theorem 18 in [5] it follows that

$$u_1(x,t)\leqslant u(x,t)\leqslant u_2(x,t),$$

and hence

$$v_1(x,t)\leqslant v(x,t)\leqslant v_2(x,t)\qquad \text{for } x\leqslant x_0,\ s\leqslant t\leqslant s+\delta. \tag{20}$$

According to the definition of v_i $(i=1,2)$ and (20) we have

$$\zeta_1-(p+\varepsilon)(t-s)\leqslant \zeta_1(t)\leqslant \zeta_1-(p-\varepsilon)(t-s)\qquad \text{for } s\leqslant t\leqslant s+\delta,$$

which gives (15) for $i=1$.

The proof of (15) for $i=2$ is similar and we omit it.

Unfortunately, we do not know by our theorem what assumptions on v_0 guarantee boundedness of $\partial^2 v/\partial x^2$ in $P[u]$. The example given in [2] shows that $|\partial^2 v/\partial x^2|$ could be unbounded even if $v_0\in C^\infty([a_1,a_2])$. If v_0' is absolutely continuous and if

$$\operatorname*{ess\,inf}_{[a_1,a_2]} v_0''(x)\geqslant -a,$$

then it can be shown that $\partial^2 v/\partial x^2 \geqslant -a$ in $P[u]$ (the proof is the same as that of Lemma 2 in [3]). If, additionally, v_0 is concave downward on $[a_1, a_2]$, then from Theorem 8 in [4] it follows that v is also concave downward as a function of x on $[\zeta_1(t), \zeta_2(t)]$ for each $t > 0$. So, in this case, $-a \leqslant \partial^2 v/\partial x^2 \leqslant 0$ in $P[u]$.

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Reçu par la Rédaction le 24. 11. 1977