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ASSOCIATE AND PSEUDOASSOCIATE SETS IN LCA GROUPS

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0. Introduction. Let G be a compact Abelian group with dual $\hat{G}$. Let $K \subseteq G$ be compact and let $\Lambda \subseteq \hat{G}$. Then K and Λ are *associate* if for every measure $\mu \in M(G)$ there exists a measure $\nu \in M(K)$ such that the Fourier-Stieltjes transforms of μ and ν agree on Λ: $\hat{\mu}(\lambda) = \hat{\nu}(\lambda)$ for all $\lambda \in \Lambda$. The sets K and Λ are *pseudoassociate* if the preceding holds for pseudomeasures in place of measures. (We recall that $S \in PM(K)$ if $\hat{S} \in l^{\infty}(G)$ and $\langle \hat{S}, f\rangle = 0$ whenever $f \in L^1(\hat{G})$ and $\hat{f}$ vanishes in a neighborhood of K.)

In this paper we construct a pair $K \subseteq \boldsymbol{T}$ and $\Lambda \subseteq \boldsymbol{Z}$ which are not associate, but which are pseudoassociate. In Section 1 such a pair is produced for a certain Cantor group. In Section 2 a (standard) tensor method is used to transfer the example to $\boldsymbol{T}$ and $\boldsymbol{Z}$. The particular example given in the sequel has the further property that Λ is a Sidon set. This is merely a technical convenience, and it seems likely that a second tensoring could be used to obtain a non-Sidon Λ' which was not associated with K' and was pseudoassociated.

This result answers a question of Professor S. Hartman (private communication). We are grateful to him for many stimulating letters.

We use freely the results and notation of the book of Rudin [2]. For more about tensor algebras, see [3].

1. A non-associate, pseudoassociate pair. Let Z_n denote the group of n-th roots of unity and let $x_n' \in Z_n$ be a fixed primitive n-th root of unity. Let $\gamma_{n,j}' \in \hat{Z}_n$ be such that

$$(x_n', \gamma_{n,j}') = \exp\left(\frac{2\pi i j}{n}\right).$$

Let $G_n = (Z_n)^{n-1}$, $x_n = (x_n, \ldots, x_n)$, and

$$\gamma(n, j) = (\underbrace{0, \ldots, 0}_{j-1 \text{ times}}, \gamma_{n,j}', 0, \ldots, 0) \in G_n, \quad 1 \leqslant j < n.$$

Set $K_n = \{jx_n : j = 0, 1, \ldots, n-1\}$, so $K_n \simeq Z_n$. Let

$$\Lambda_n = \{\gamma(n, j) : j = 1, \ldots, n-1\} \quad \text{and} \quad \Lambda'_n = \Lambda_n \cup \{0\}.$$

Then

$$\Lambda'_n|_{K_n} = \hat{K}_n, \tag{1}$$

as spaces of functions. It follows at once from (1) that

$$S \in PM(K_n) \text{ implies } \|S\|_{PM} = \sup_{\lambda \in \Lambda'_n} |\hat{S}(\lambda)|. \tag{2}$$

We now define our Cantor group and sets K and Λ. Set

$$K = \prod_{n=2}^{\infty} K_n, \quad G = \prod_{n=2}^{\infty} G_n, \quad \text{and} \quad \Lambda = \bigcup_{n=2}^{\infty} \Lambda_n.$$

LEMMA A. $PM(K)\hat{}\,|_\Lambda = l^\infty(\Lambda) = B(\Lambda)$.

Proof. The equality $B(\Lambda) = l^\infty(\Lambda)$ holds, since Λ is independent and hence a Sidon set (see [1] and [2]).

For the other equality, let $f \in l^\infty(\Lambda)$ have supremum at most one. For $n = 2, 3, \ldots$ define $f_n : \Lambda'_n \to C$ by

$$f_n(\lambda) = f(\lambda), \lambda \in \Lambda_n, \quad \text{and} \quad f_n(0) = 1. \tag{3}$$

Then (1) and (2) imply that there exists $S_n \in PM(K_n)$ with $\hat{S}_n = f_n$, and $\|S_n\|_{PM} = 1$. Let S be the (weak* limit of the finite subproducts of the) infinite product

$$S = \mathop{\text{X}}_{n=2}^{\infty} S_n.$$

Then $S \in PM(K)$ and (3) implies $\hat{S} = f$ on Λ. This proves Lemma A.

LEMMA B. $M(K)\hat{}\,|_\Lambda \neq l^\infty(\Lambda)$.

Proof. We use (1). Suppose that every element of $l^\infty(\Lambda)$ is the restriction of the Fourier-Stieltjes transform of a measure on K. Then, by the open mapping theorem, there exists a constant $A > 0$ such that if $f \in l^\infty(\Lambda)$, then there exists $\mu \in M(K)$ with

$$\hat{\mu}|_\Lambda = f \quad \text{and} \quad \|\mu\|_{M(K)} \leqslant A\|f\|_\infty.$$

Fix $n \geqslant 2$. Then for each $\mu \in M(K)$ we have the representation

$$\mu = \sum_{x \in K_n} \delta_x \times \mu_{x,n}, \quad \text{where } \mu_{x,n} \in M\Big(\prod_{m \neq n} K_m\Big), \tag{4}$$

and

$$\|\mu\| = \sum \|\mu_{x,n}\|. \tag{5}$$

Of course, for $\lambda \in \Lambda_n'$ we have

$$\hat{\mu}(\lambda) = \sum (\lambda, x)\, \hat{\mu}_{x,n}(0) = \sum \hat{\mu}_{x,n}(0)\hat{\delta}_x(\lambda). \tag{6}$$

Hence, by (1) and our choice of A, we see that every function bounded by one on $\hat{Z}_n \setminus \{0\}$ is the restriction of the Fourier-Stieltjes transform of a measure on K_n of the form given by (6), whose norm is at most A (using (5)). But it is well known that every Sidon set of Sidon constant A can contain at most $B \log N$ terms of an arithmetic progression of length N, where B depends only on A (see [1]). The preceding shows that every set $\hat{Z}_n \setminus \{0\}$ is a Sidon set of constant A, so we have a contradiction.

This completes the proof of Lemma B.

2. Tensor method. Let G, K, and Λ be as in Section 1. Let

$$V(G, G) = C(G) \hat{\otimes} C(G)$$

be the usual tensor product (greatest cross norm) algebra. Let

$$K_1 = \{(x, y) \in G \times G : x - y \in K\},$$

and let

$$\Lambda_1 = \{f_\lambda \in V(G, G) : f_\lambda(x, y) = (\lambda, x - y), \lambda \in \Lambda\}.$$

LEMMA C. *If $F: \Lambda_1 \to C$ is bounded, then there exists $S \in V(G, G)^*$ such that*

$$\operatorname{support} S \subseteq K_1 \quad \textit{and} \quad \langle S, f_\lambda \rangle = F(f_\lambda), \lambda \in \Lambda.$$

LEMMA D. *There exists a bounded function $F: \Lambda_1 \to C$ such that no measure $\mu \in M(K_1)$ has*

$$\int f_\lambda d\mu = F(f_\lambda) \quad \textit{for all } \lambda \in \Lambda.$$

Proof of Lemmas C and D. The map M: $C(G) \to C(G \times G)$ defined by

$$Mf(x, y) = f(x - y)$$

embeds $A(G)$ isometrically in $V(G, G)$, and the map P: $C(G \times G) \to C(G)$ given by

$$Pf(x) = \int f(x - y, y)\, dy$$

maps $V(G, G)$ onto $A(G)$. (These facts are in [3].) Hence P^* maps $PM(G)$ isometrically into $V(G, G)^*$ and if $S \in PM(K)$, then $P^* S$ has support K_1. Of course, $f_\lambda = M\lambda$.

These observations, and Lemmas A and B now complete the proof of Lemmas C and D. (It is perhaps unnecessary to add that M^* maps $M(K_1)$ onto $M(K)$.)

Now let H_1 and H_2 be disjoint Cantor sets in $\boldsymbol{T}$ whose union is a Kronecker set [2]. Let $\varphi_j: H_j \to G$ be surjective homeomorphisms such that

$$\varphi = (\varphi_1, \varphi_2): H_1 \times H_2 \to G \times G$$

induces an algebra isometric isomorphism of $V(G, G)$ with $V(H_1, H_2)$. Let g_λ be the image of f_λ under this map ($\lambda \in \Lambda$) and let $\Lambda_2 = \{g_\lambda : \lambda \in \Lambda\}$.

By the definition of f_λ, we know that g_λ is the product of two unimodular functions, so, by the Kronecker property of $H_1 \cup H_2$, there exists, for each $\lambda \in \Lambda$, an element $\gamma_\lambda \in \boldsymbol{Z}$ such that

(7) $$\|\gamma_\lambda|_{H_1} \otimes \gamma_\lambda|_{H_2} - g\|_{V(G,G)} < \frac{1}{4}.$$

Then (7), Lemmas C and D, and easy summation arguments (see [3], for example) imply

LEMMA C′. *If F: $\{\gamma_\lambda : \lambda \in \Lambda\} \to \boldsymbol{C}$ is bounded, then there exists $S \in V(H_1, H_2)^*$ such that*

$$\langle S, \gamma_\lambda|_{H_1} \otimes \gamma_\lambda|_{H_2}\rangle = F(\gamma_\lambda) \quad \textit{for all } \lambda \in \Lambda.$$

LEMMA D′. *There exists a bounded function F: $\{\gamma_\lambda : \lambda \in \Lambda\} \to \boldsymbol{C}$ such that no measure $\mu \in M\big((\varphi_1, \varphi_2)^{-1} K_1\big)$ has*

$$\int \gamma_\lambda|_{H_1} \otimes \gamma_\lambda|_{H_2} d\mu = F(\gamma_\lambda) \quad \textit{for all } \lambda \in \Lambda.$$

Let $\pi: H_1 \times H_2 \to H_1 + H_2$ be given by $\pi(x, y) = x + y$. Set

$$K_3 = \pi(\varphi_1, \varphi_2)^{-1}(K_1) \subseteq H_1 + H_2 \quad \text{and} \quad \Lambda_3 = \{\gamma_\lambda : \lambda \in \Lambda\}.$$

Then

(8) $$PM(K_3) = \pi^* \{S \in V(H_1, H_2)^* : \text{support} S \subseteq K_2\},$$

where $K_2 = (\varphi_1, \varphi_2)^{-1}(K_1)$. Also, since $\gamma_\lambda \in \boldsymbol{Z}$, we have

(9) $$\gamma_\lambda \circ \pi = \gamma_\lambda|_{H_1} \otimes \gamma_\lambda|_{H_2}.$$

A dull collection of computations now show that K_3 and Λ_3 are pseudoassociate and are not associate.

Added in proof. The second part of this note, in which an associate non-pseudoassociate pair is constructed, will appear in this journal.

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