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ON A METRICAL THEOREM IN GEOMETRY OF CIRCLES

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The radical center theorem (see [1]) says:

The common chords of three circles taken in pairs are concurrent.

With the help of this theorem we can prove the following

THEOREM. *Draw three circles each of which cuts the other two. Let the pairs of circles meet in points* A_1, A_2; B_1, B_2; C_1, C_2. *Then* $\overline{B_1C_2}\cdot\overline{C_1A_2}\cdot\overline{A_1B_2} = \overline{C_1B_2}\cdot\overline{A_1C_2}\cdot\overline{B_1A_2}$ *holds.*

Proof. We put aside the trivial case where $A_1A_2 // B_1B_2 // C_1C_2$. Then, by the radical center theorem A_1A_2, B_1B_2, C_1C_2 are concurrent. Let the point of concurrence be denoted by O. Since B_1, C_1, B_2, C_2 are concyclic, the triangles OB_1C_2, OC_1B_2 are similar. Hence we have

$$\frac{\overline{B_1C_2}}{\overline{OB_1}} = \frac{\overline{C_1B_2}}{\overline{OC_1}}. \tag{1}$$

Since C_1, A_1, C_2, A_2 are concyclic, the triangles OC_1A_2, OA_1C_2 are similar. Hence we have

$$\frac{\overline{C_1A_2}}{\overline{OC_1}} = \frac{\overline{A_1C_2}}{\overline{OA_1}}. \tag{2}$$

Since A_1, B_1, A_2, B_2 are concyclic, the triangles OA_1B_2, OB_1A_2 are similar. Hence we have

$$\frac{\overline{A_1B_2}}{\overline{OA_1}} = \frac{\overline{B_1A_2}}{\overline{OB_1}}. \tag{3}$$

Multiplying (1), (2), (3) side by side, we have

$$\overline{B_1C_2}\cdot\overline{C_1A_2}\cdot\overline{A_1B_2} = \overline{C_1B_2}\cdot\overline{A_1C_2}\cdot\overline{B_1A_2}, \qquad \text{q.e.d.}$$

The purpose of this note is to give an application of the above theorem to a proof of the "only if" part of the following principle of circle-transformation of a linear rational function (see [2]):

Suppose that $f = f(z)$ ($\not\equiv$ const) is meromorphic in $|z| < +\infty$. Then $w = f(z)$ transforms circles in the z-plane onto circles in the w-plane, including straight lines among circles, if and only if f is a linear rational function.

We may now prove the above principle. Since $f \not\equiv$ const, the domain D where f is regular and $f'(z) \neq 0$ is not empty. Let x be an arbitrarily fixed point belonging to D. Since $f'(x) \neq 0$, f maps a sufficiently small circular neighborhood of x one-to-one conformally onto a neighborhood of $f(x)$. Let this circular neighborhood of x be $|z-x| < r$ where r is a positive real constant.

Consider an equilateral triangle with circumcenter at x. Let O, A, B, C represent the complex numbers $x, x+y, x+\omega y, x+\omega^2 y$ where y is complex with $|y| < (\sqrt{3}-1)r$ and $\omega = \exp\left(\frac{2\pi i}{3}\right)$. Furthermore, let L, M, N be the middle points of the sides BC, CA, AB. The following sets of four points are all concyclic: B, C, M, N; B, L, O, N; C, L, O, M. Let the circle passing through B, C, M, N be O_1, the circle passing through B, L, O, N be O_2 and the circle passing through C, L, O, M be O_3. Since $|y| < (\sqrt{3}-1)r$, after some calculations we see that the above three circles are contained in $|z-x| < r$. Since f is univalent in $|z-x| < r$, $f(O_1), f(O_2), f(O_3)$ are univalent maps of O_1, O_2, O_3, respectively. By hypothesis $f(O_1), f(O_2), f(O_3)$ are all circles. By the above argument each of these three circles cuts the other two and if we put $S' = f(S)$ for any point S, then the pairs of circles meet in the points: O', L'; C', M'; B', N'. Hence, by the above theorem,

$$\overline{B'M'} \cdot \overline{L'N'} \cdot \overline{C'O'} = \overline{C'N'} \cdot \overline{B'O'} \cdot \overline{L'M'}, \tag{4}$$

where

$$\overline{B'M'} = \left| f(x+\omega y) - f\left(x - \frac{1}{2}\omega y\right) \right|,$$

$$\overline{L'N'} = \left| f\left(x - \frac{1}{2}y\right) - f\left(x - \frac{1}{2}\omega^2 y\right) \right|,$$

$$\overline{C'O'} = |f(x+\omega^2 y) - f(x)|,$$

$$\overline{C'N'} = \left| f(x+\omega^2 y) - f\left(x - \frac{1}{2}\omega^2 y\right) \right|,$$

$$\overline{B'O'} = |f(x+\omega y) - f(x)|,$$

$$\overline{L'M'} = \left| f\left(x - \frac{1}{2}y\right) - f\left(x - \frac{1}{2}\omega y\right) \right|.$$

By (4) and by the Maximum Modulus Theorem we have

(5)
$$\frac{\left(f(x+\omega y)-f\left(x-\frac{1}{2}\,\omega y\right)\right)\left(f\left(x-\frac{1}{2}\,y\right)-f\left(x-\frac{1}{2}\,\omega^2 y\right)\right)(f(x+\omega^2 y)-f(x))}{\left(f(x+\omega^2 y)-f\left(x-\frac{1}{2}\,\omega^2 y\right)\right)(f(x+\omega y)-f(x))\left(f\left(x-\frac{1}{2}\,y\right)-f\left(x-\frac{1}{2}\,\omega y\right)\right)} = C,$$

where C is a complex constant of modulus 1 and may or may not depend on x.

As $y\to 0$, de L'Hospital's Theorem implies

$$\frac{f(x+\omega y)-f\left(x-\frac{1}{2}\,\omega y\right)}{f(x+\omega^2 y)-f\left(x-\frac{1}{2}\,\omega^2 y\right)} \to \frac{\omega f'(x)+\frac{1}{2}\,\omega f'(x)}{\omega^2 f'(x)+\frac{1}{2}\,\omega^2 f'(x)} = \frac{1}{\omega},$$

$$\frac{f\left(x-\frac{1}{2}\,y\right)-f\left(x-\frac{1}{2}\,\omega^2 y\right)}{f(x+\omega y)-f(x)} \to \frac{-\frac{1}{2}f'(x)+\frac{1}{2}\,\omega^2 f'(x)}{\omega f'(x)} = \frac{-1+\omega^2}{2\omega},$$

$$\frac{f(x+\omega^2 y)-f(x)}{f\left(x-\frac{1}{2}\,y\right)-f\left(x-\frac{1}{2}\,\omega y\right)} \to \frac{\omega^2 f'(x)}{-\frac{1}{2}f'(x)+\frac{1}{2}\,\omega f'(x)} = \frac{2\omega^2}{-1+\omega}.$$

Hence, by (5),

(6)
$$C = -\omega^2.$$

By (5) and (6) we have in $|y|<(\sqrt{3}-1)r$

(7)
$$\left(f(x+\omega y)-f\left(x-\frac{1}{2}\,\omega y\right)\right)\left(f\left(x-\frac{1}{2}\,y\right)-f\left(x-\frac{1}{2}\,\omega^2 y\right)\right)(f(x+\omega^2 y)-f(x))+$$
$$+\omega^2\left(f(x+\omega^2 y)-f\left(x-\frac{1}{2}\,\omega^2 y\right)\right)(f(x+\omega y)-f(x))\times$$
$$\times\left(f\left(x-\frac{1}{2}\,y\right)-f\left(x-\frac{1}{2}\,\omega y\right)\right) = 0.$$

Differentiating both sides of (7) five times with respect to y and putting $y=0$, we infer

(8)
$$2f'(x)f'''(x)-3f''(x)^2 = 0.$$

Since x is an arbitrarily fixed point belonging to D, by (8) we have in D

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 0. \tag{9}$$

The left-hand side of (9) is the Schwarz's derivative of f. Solving (9), we see that f is a linear rational function.

REFERENCES

[1] E. A. Maxwell, *Geometry for advanced pupils*, Oxford 1957, p. 20-21.
[2] Z. Nehari, *Conformal mapping*, New York - Toronto - London 1952, p. 160.

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