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A SUFFICIENT CONDITION FOR A VARIETY TO HAVE THE AMALGAMATION PROPERTY

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A class K of algebras has the *amalgamation property* [4] if for each pair $\mathscr{A}_1$, $\mathscr{A}_2$ of algebras in K, if $\mathscr{B} \in K$ and i_1, i_2 are imbeddings of $\mathscr{B}$ into $\mathscr{A}_1$ and $\mathscr{A}_2$, then there is a member $\mathscr{C}$ of K and imbeddings f_1, f_2 of $\mathscr{A}_1$, $\mathscr{A}_2$ into $\mathscr{C}$ such that $i_1 f_1 = i_2 f_2$. Fajtlowicz [3] proposed problem P 644: does an equationally complete variety K have the amalgamation property? We provide a partial solution to this question.

A *quasi-variety* is a class of algebras closed under direct product and subalgebra. If K is a class of algebras let T_k be the first order theory of the infinite members of K. A first order theory T is *model complete* if, for models $\mathscr{A}$ and $\mathscr{B}$ of T, $\mathscr{A}$ a submodel of $\mathscr{B}$ implies $\mathscr{A}$ is an elementary submodel of $\mathscr{B}$.

A class K of algebras has the *non-trivial amalgamation property* if $\mathscr{A}_1, \mathscr{A}_2 \in K$, $\mathscr{B} \in K$, $\mathscr{B}$ has at least two elements, and i_1, i_2 are imbeddings of $\mathscr{B}$ into $\mathscr{A}_1$, $\mathscr{A}_2$ implies that there exist $\mathscr{C}$ in K and imbeddings f_1, f_2 of $\mathscr{A}_1$, $\mathscr{A}_2$ into $\mathscr{C}$ such that $i_1 f_1 = i_2 f_2$.

THEOREM 1. *If K is a class of algebras closed under direct power and elementary extension, and T_k is model complete, then K has the non-trivial amalgamation property.*

Proof. Let $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{B}$ be elements of K such that $\mathscr{B}$ has at least two elements. Let i_1, i_2 be imbeddings of $\mathscr{B}$ into $\mathscr{A}_1$ and $\mathscr{A}_2$. Then the direct powers $(\mathscr{B})^{\aleph_0}$, $(\mathscr{A}_1)^{\aleph_0}$ and $(\mathscr{A}_2)^{\aleph_0}$ are all models of T_k. Moreover, $(\mathscr{B})^{\aleph_0}$ is an elementary submodel of $(\mathscr{A}_1)^{\aleph_0}$ and $(\mathscr{A}_2)^{\aleph_0}$. Hence (e.g. [6], 1.2 b) there is an elementary extension $\mathscr{C}$ of $(\mathscr{A}_1)^{\aleph_0}$ and $(\mathscr{A}_2)^{\aleph_0}$. But then $\mathscr{C}$ is in K and the composition of the diagonal map of $\mathscr{A}_i$ into $(\mathscr{A}_i)^{\aleph_0}$ with the elementary imbedding of $(\mathscr{A}_i)^{\aleph_0}$ into $\mathscr{C}$ provides the required f_i for $i = 1, 2$.

The following strengthening of Theorem 1 derives from an observation of A. H. Lachlan. There is an example after Theorem 1 of [1] showing the necessity of replacing direct power in Theorem 1 by direct product in Theorem 2.

THEOREM 2. *If K is a class of algebras closed under direct product and elementary extension and T_k is model complete, then K has the amalgamation property.*

Proof. Applying Theorem 1, it suffices to show that if $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{B} \in K$, $\mathscr{B}$ has one element, and i_1, i_2 are injections of $\mathscr{B}$ into $\mathscr{A}_1$ and $\mathscr{A}_2$ there exists $\mathscr{C} \in K$ and injections f_1, f_2 of $\mathscr{A}_1$, $\mathscr{A}_2$ into $\mathscr{C}$ such that $i_1 f_1 = i_2 f_2$.

Let $\mathscr{C} = \mathscr{A}_1^{\aleph_0} \times \mathscr{A}_2^{\aleph_0}$. Let $f_1(a) = \langle a, a, \ldots, i_2(b), i_2(b), \ldots \rangle$ for $a \in |\mathscr{A}_1|$ and let $f_2(a) = \langle i_1(b), i_1(b), \ldots, a, a, \ldots \rangle$ for $a \in |\mathscr{A}_2|$, where b is the unique element of $|\mathscr{B}|$. Noticing that b must be an idempotent, it follows that f_1 and f_2 are the required maps.

Now let K be a variety (indeed any elementary class of algebras closed under the unions of chains) and suppose K is categorical in some infinite power α (i.e. all members of K with power α are isomorphic). Lindstrom [5] proved that a first order theory, which has no finite models, is categorical in some infinite power, and whose class of models is closed under unions of chains is model complete. Hence T_k is model complete. We have

THEOREM 3. *If a variety K is categorical in some infinite power, then K has the amalgamation property.*

Obviously, applying the Łoś-Vaught test if the variety K is categorical in some infinite power, it is equationally complete.

These methods are used to prove Theorem 1 of [1]:

If the theory of infinite models of a universal Horn class is model complete, it is quantifier eliminable.

Using terminology of Abraham Robinson we can derive that result from Theorem 2 (replacing variety by universal Horn theory in the hypothesis). The terms used in the following argument can be found in [2]. If T_K is model complete T_K is clearly a model companion to T'_K (the theory of all models of K). Lemma 2.1 (due to Bers) of [2] asserts that if a theory S has a model companion, then S has a model completion if and only if S has the amalgamation property. Hence, by Theorem 2, T_K is a model completion of T'_K. Robinson proved (see [7], Theorem 13.2) that the model completion of a universal theory is quantifier eliminable. Hence T_K is quantifier eliminable.

Utilizing Theorem 3 of [1] we may improve Theorem 3 to: if K is a quasi-variety and T_K is complete, then K has the amalgamation property.

REFERENCES

[1] J. T. Baldwin and A. H. Lachlan, *On universal Horn theories categorical in some infinite power*, Algebra Universalis, to appear.
[2] P. Eklof and G. Sabbagh, *Model-completions and modules*, Annals of Mathematical Logic 2 (1971), p. 251-295.
[3] S. Fajtlowicz, *P 644*, Colloquium Mathematicum 19 (1969), p. 334.
[4] B. Jónsson, *Homogeneous universal relational systems*, Mathematica Scandinavica 8 (1960), p. 137-142.
[5] P. Lindstrom, *On model completeness*, Theoria 30 (1964), p. 183-196.
[6] M. Morley and R. Vaught, *Homogeneous universal models*, Mathematica Scandinavica 11 (1962), p. 37-57.
[7] G. E. Sacks, *Saturated model theory*, Reading 1972.

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