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ON DIFFERENTIAL SUBSPACES OF CARTESIAN SPACE

BY

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1. Introduction. In 1967 Sikorski (cf. [4]) introduced the concept of a differential space as a generalization of differentiable manifold. Independently, Mac Lane (cf. [1]) introduced the same concept in his lectures on foundations of mechanics. Any subset of the set of all points of a differential space may be regarded as a differential space with the induced differential structure. More exactly, if C is a set of real functions on a set M, then we have the weakest topology τ_C such that all functions in C are continuous on (M, τ_C). For any $A \subset M$ one defines (cf. [4]) the set C_A of all functions $\beta\colon A \to \boldsymbol{R}$ such that for any $p \in A$ there exist $\alpha \in C$ and U in τ_C, $p \in U$, with $\beta | A \cap U = \alpha | A \cap U$. In particular, we can take $A = M$. We define (cf. [1]) $\mathrm{sc}\, C$ as the set of all real functions of the form $\omega(\alpha_1(\cdot), \ldots, \alpha_s(\cdot))$, where $\alpha_1, \ldots, \alpha_s$ are in C, ω is of class C^∞ on $\boldsymbol{R}^s$, and s is any positive integer. We have $C \subset C_M$ and $C \subset \mathrm{sc}\, C$. The set C satisfying the equalities $C_M = C = \mathrm{sc}\, C$ is said to be a *differential structure* on M and (M, C) is called a *differential space*. Then, for any $A \subset M$ we have a differential space (A, C_A).

In the Cartesian space $\boldsymbol{R}^n$ we have the natural differential structure E_n, denoted shortly by E, which is composed of all real functions of class C^∞ on $\boldsymbol{R}^n$. If M is a subset of $\boldsymbol{R}^n$, then we may consider M together with the differential structure E_M. In particular, in the theory of differential spaces, the concept of tangent space at each point of the space is considered.

Let (M, C) be a differential space and $p \in M$. Following Sikorski, we denote by $(M, C)_p$ the tangent space of (M, C) at p. The vectors of this space are linear mappings v of C into $\boldsymbol{R}$ such that $v(\alpha\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha)$ for $\alpha, \beta \in C$. In the natural way, C is regarded as a linear space.

There is a natural isomorphism (cf., e.g., [3] and [2]) between the vector space $(M, C)_p$ and the vector space $T_p(M, C)$, dual to the linear ring $C(p, 0)/C^2(p, 0)$; here $C(p, 0)$ is the ideal of all germs at the point p of functions $\alpha \in C$, $\alpha(p) = 0$, in the linear ring of all germs at p of functions belonging to C (with respect to topology τ_C) and $C^2(p, 0)$ stands for the square of this ideal, i.e. $C^2(p, 0)$ is the ideal of all germs

$\xi_1\eta_1 + \ldots + \xi_s\eta_s$, where $\xi_1, \eta_1, \ldots, \xi_s, \eta_s$ are in $C(p, 0)$ and s is any positive integer.

For any ξ in $C(p, 0)$ and for any subset A of M, $p \in A$, we can define ξ_A as the germ in $C_A(p, 0)$ such that $(\alpha|A, p) \in \xi_A$, where $(\alpha, p) \in \xi$. We denote by $[\xi]_C$ the coset of ξ in $T_p^*(M, C)$, where $T_f^*(M, C)$ stands for $C(p, 0)/C^2(p, 0)$. Similarly, for μ in $C_A(p, 0)$ the coset of μ in $T_p^*(A, C_A)$ will be denoted by $[\mu]_{C_A}$. It is easy to check the following

PROPOSITION 1.1. *If $p \in A \subset M$, then we have the epimorphisms*

$$\xi \mapsto \xi_A: C(p, 0) \to C_A(p, 0), \qquad \xi \mapsto [\xi]_C: C(p, 0) \to T_p^*(M, C),$$

$$\mu \mapsto [\mu]_{C_A}: C_A(p, 0) \to T_p^*(A, C_A)$$

such that the diagram

$$\begin{array}{ccc} C(p, 0) & \longrightarrow & C_A(p, 0) \\ \downarrow & & \downarrow \\ T_p^*(M, C) & \longrightarrow & T_p^*(A, C_A) \end{array}$$

is commutative. In particular, $\dim T_p^*(A, C_A) \leqslant \dim T_p^*(M, C)$.

For any $M \subset \mathbf{R}^n$ we denote by T_p^*M the vector space $T_p^*(M, E_M)$. It is easy to check that $\dim T_p^*\mathbf{R}^n = n$.

PROPOSITION 1.2. *If $p \in M \subset \mathbf{R}^n$, then* $\dim(M, E_M)_p = \dim T_p^*M \leqslant n$.

Proof. By Proposition 1.1 we have $\dim T_p^*M \leqslant \dim T_p^*\mathbf{R}^n = n$. To complete the proof it suffices to take an isomorphism between the finite-dimensional vector spaces $T_p^*(M, E_M)$ and $T_p(M, E_M)$, and the natural isomorphism (see [3] and [4]) between $T_p(M, E_M)$ and $(M, E_M)_p$.

We recall (cf. [1], [4]) that f maps *smoothly* (M, C) into (N, D), which we write

$$f: (M, C) \to (N, D), \tag{1.1}$$

if f maps the set M into the set N and for any $\beta \in D$ the real function $\beta \circ f$ is in C. For every point $p \in M$ the mapping (1.1) induces the linear mapping

$$f_{*p}: (M, C)_p \to (N, D)_{f(p)}$$

of tangent spaces, called the *tangent mapping* to (1.1) at the point p, where for any v in $(M, C)_p$ and for any $\beta \in D$ we set $f_*(v)(\beta) = v(\beta \circ f)$.

In particular, if $M \subset \mathbf{R}^n$, then the identity mapping

$$i_M: (M, E_M) \to (\mathbf{R}^n, E) \tag{1.2}$$

induces the tangent mapping

$$i_{M*p}\colon (M, E_M)_p \to (R^n, E)_p. \tag{1.3}$$

If $w \in (R^n, E)_p$ is of the form $w^i(\partial/\partial x_i)_p$, where $(\partial/\partial x_i)_p$ denotes the partial derivative at p of the functions in E, then

$$w \mapsto \hat{w}, \quad \text{where } \hat{w} = (w^1, \ldots, w^n), \tag{1.4}$$

establishes an isomorphism between $(R^n, E)_p$ and the vector space R^n. The hyperplane M_p derived from $(M, E_M)_p$ by taking the mappings i_{M*p}, $w \mapsto \hat{w}$, and the translation $q \mapsto p+q$ will be called the *tangent hyperplane* to the differential space (M, E_M) (or shortly, to the set M) at the point p.

2. The hyperplane of directions at a point. We say that the sequences

$$p_1, p_2, \ldots \quad \text{and} \quad p'_1, p'_2, \ldots \tag{2.1}$$

of points belonging to the subset M of R^n *determine a direction l of M at p* if the sequences (2.1) tend to p, $p_k \neq p'_k$ for $k = 1, 2, \ldots$, and

$$(p'_k - p_k)|p'_k - p_k|^{-1} \to l \quad \text{as } k \to \infty. \tag{2.2}$$

Obviously, $|l| = 1$. A direction l defined by some sequences (2.1) of points in M is said to be a *direction of M at p*. The smallest of all hyperplanes containing p as well as all points $p+l$, where l is any direction of M at p, will be called the *hyperplane of directions of M at p* and denoted by M_p^d.

PROPOSITION 2.1. *For every $p \in M$ we have $M_p^d \subset M_p$.*

Proof. Let l be any direction of M at p. We have some sequences (2.1) tending to p such that (2.2) holds. For any real function a in E_M there exist a neighbourhood U of p and a function β in E such that $a|M\cap U = \beta|M\cap U$. There exist real functions β_i of class C^∞ on $R^n \times R^n$ such that for every $q = (q^1, \ldots, q^n)$ and $q' = (q'^1, \ldots, q'^n)$ in R^n we have

$$\beta(q') - \beta(q) = \beta_i(q', q)(q'^i - q^i), \quad \beta_i(p, p) = (\partial/\partial x_i)_p(\beta).$$

Hence

$$\begin{aligned}\big(a(p'_k) - a(p_k)\big)|p'_k - p_k|^{-1} &= \big(\beta(p'_k) - \beta(p_k)\big)|p'_k - p_k|^{-1} \\ &= \beta_i(p'_k, p_k)(p'^i_k - p^i_k)|p'_k - p_k|^{-1} \to (\partial/\partial x_i)_p(\beta)l^i \quad \text{as } k \to \infty,\end{aligned}$$

where $p_k = (p_k^1, \ldots, p_k^n)$, $p'_k = (p'^1_k, \ldots, p'^n_k)$, $l = (l^1, \ldots, l^n)$. The number $(\partial/\partial x_i)_p(\beta)l^i$ is independent of the choice of the function β. Then we can denote this number by $l_p(a)$, and we get a vector in M_p. It is easy to check that the corresponding vector $\big(i_{M*p}(l_p)\big)\hat{}$ is equal to l. Thus $p+l = p+$ $+\big(i_{M*p}(l_p)\big)\hat{}$ is a point in M_p, which completes the proof.

As an immediate consequence of Proposition 2.1 we have

PROPOSITION 2.2. *A point p in M is a cluster point of M if and only if* $\dim M_p \geqslant 1$.

3. The local description of a subset of R^n. Let $p \in M \subset R^n$. We will consider any hyperplane H containing the hyperplane of directions of M at the point p and we will examine the local situation of M with respect to H. By $B(p, r)$ we denote the ball with the center p and the radius r. For any point q of H the orthogonal complement of H passing through q will be denoted by $H_q^{\perp}$. The set of all $r > 0$ such that for every q in $H \cap B(p, r)$ the set

$$M \cap H_q^{\perp} \cap B(p, r) \tag{3.1}$$

has at most one element will be denoted by $M(p, H)$. For any $r \in M(p, H)$ we consider the mapping p_{rM}^H assigning to every $q \in H \cap B(p, r)$ such that (3.1) is non-empty the only element $p_{rM}^H(q)$ contained in the set (3.1).

The following proposition is obvious.

PROPOSITION 3.1. *For any $r \in M(p, H)$ the set $M \cap B(p, r)$ coincides with the set of all values of the mapping p_{rM}^H.*

Now, we prove the following

PROPOSITION 3.2. *If $M_p^d \subset H$, then the following conditions are satisfied:*

(i) $\sup M(p, H) > 0$ *and* $M(p, H) = (0; \sup M(p, H)\rangle$;

(ii) *there exists an $s \in M(p, H)$ such that for any $r \in (0; s)$ the mapping p_{rM}^H is uniformly continuous with respect to its domain.*

Proof. (i) Assuming that $M(p, H) = \emptyset$, we have the sequence of points $q_k \in H \cap B(p, 1/k)$ such that each of the sets $M \cap H_{q_k}^{\perp} \cap B(p, 1/k)$ has two different points p_k' and p_k. Then we have $p_k \to p$ and $p_k' \to p$ as $k \to \infty$, and the directions $l_k = (p_k' - p_k)|p_k' - p_k|^{-1}$ are parallel to $H_{q_k}^{\perp}$. Consequently, they are orthogonal to H. Passing to subsequences, if necessary, we may assume that $l_k \to l$ as $k \to \infty$. Then we get the direction l of M at p, being orthogonal to H. Thus, l is orthogonal to M_p^d, a contradiction.

Now, we set $r_p = \sup M(p, H)$. Thus we have $r_p > 0$. The inclusion $M(p, H) \subset (0; r_p\rangle$ is obvious. First, let $0 < t < r_p$. Then there exists an $r \in M(p, H)$, $t < r < r_p$. Then, for every $q \in H \cap B(p, r)$ the set (3.1) has at most one element. Assuming that $r_p \notin M(p, H)$, we should find a point $q \in H \cap B(p, r_p)$ such that the set $M \cap H_q^{\perp} \cap B(p, r_p)$ has two different points q_1 and q_2. Taking r such that

$$\max\{|p - q|, |p - q_1|, |p - q_2|\} < r < r_p,$$

we get $q \in H \cap B(p, r)$, $q_1 \neq q_2$, and $q_1, q_2 \in M \cap H_q^{\perp} \cap B(p, r)$, a contradiction.

(ii) In the opposite case there should exist a sequence of positive numbers r_k, two sequences of points q_k and q'_k belonging to the domain of the mapping $p^H_{r_kM}$ such that $r_k \to 0$ as $k \to \infty$, $|q'_k - q_k| < r_k \varepsilon_k$ and $|p_k - p'_k| \geq \varepsilon_k$, where $p_k = p^H_{r_kM}(q_k)$ and $p'_k = p^H_{r_kM}(q'_k)$. Then we have

$$p_k \neq p'_k, \quad p_k \in M \cap H^{\perp}_{q_k} \cap B(p, r_k) \text{ and } p'_k \in M \cap H^{\perp}_{q'_k} \cap B(p, r_k).$$

Thus, $p_k - q_k \perp H$ and $p'_k - q'_k \perp H$. Passing to subsequences, if necessary, we may assume that $l_k = (p'_k - p_k)|p'_k - p_k|^{-1} \to l$ as $k \to \infty$. Obviously, we have

$$l_k + (q_k - q'_k)|p'_k - p_k|^{-1} = (p'_k - q'_k)|p'_k - p_k|^{-1} + (q_k - p_k)|p'_k - p_k|^{-1} \perp H$$

and

$$|q_k - q'_k|\,|p_k - p'_k|^{-1} < r_k \varepsilon_k / \varepsilon_k = r_k.$$

Hence, it follows that

$$H \perp l_k + (q_k - q'_k)|p'_k - p_k|^{-1} \to l \quad \text{as } k \to \infty.$$

Therefore, l should be a direction of M orthogonal to H at p, and so orthogonal to M^d_p, which is impossible.

From Proposition 2.1 it follows that the tangent hyperplane M_p may be regarded as H in Proposition 3.2, which will be useful in the next section.

4. Differentiability of the local descriptions. We have defined the mappings p^H_{rM} which describe the set M near to the point p with respect to the hyperplane H. In this section we assume that this hyperplane H coincides with the tangent hyperplane M_p to the set M at the point p. Then the mapping p^H_{rM}, where $H = M_p$, will be denoted shortly by p_{rM}. Now, we will prove the main theorem of the paper.

THEOREM 4.1. *If $p \in M \subset \mathbf{R}^n$ and* $\dim M_p \geq 1$, *then there exists a mapping f satisfying the following conditions:*

(i) *the domain of the mapping f is contained in M_p and dense in itself;*

(ii) *there exists an $r > 0$ such that the set of all values of f coincides with $M \cap B(p, r)$;*

(iii) *for every q of the domain of f the orthogonal projection of $f(q)$ onto the tangent hyperplane M_p coincides with q;*

(iv) *the mapping f has the derivatives of all orders at each point of its domain with respect to any direction of the domain.*

Proof. Let $e_1, \ldots, e_m$ be an orthonormal basis of the linear space obtained from M_p by the translation $q \mapsto q - p$ and let $e_{m+1}, \ldots, e_n$ be an orthonormal basis of the orthogonal complement to $\mathbf{R}^n$ of this space. Let $e'_1, \ldots, e'_m$ be the basis of the vector space $(M, E_M)_p$ defined by the equalities

$$e_h = \left(i_{M*p}(e'_h)\right)^{\hat{}}, \quad h = 1, \ldots, m,$$

where i_{M*p} is the tangent mapping (1.3) to the mapping (1.2), and $\hat{}$ is defined by (1.4). We set $e^k(q) = e_k(q-p)$ for $q \in \boldsymbol{R}^n$, $k = 1, \ldots, n$. Evidently, $e^k|M$ belongs to E_M, and $e^k(p) = 0$. Let us remark that

$$e'_i(e^k|M) = i_{M*p}(e'_i)(e^k) = e^l_i(\partial/\partial x_l)_p(e^k) = e^l_i \sum_{s=1}^{n} e^s_k \delta^s_l = \sum_{l=1}^{n} e^l_i e^l_k = e_i e_k,$$

where $e_i = (e^1_i, \ldots, e^n_i)$. In other words,

$$e'_i(e^k|M) = \delta^k_i, \quad i = 1, \ldots, m;\ k = 1, \ldots, n. \tag{4.1}$$

Denote by $\bar{e}^k$ the elements of the vector space $T^*_p(M, E_M)$ such that $(e^k|M, p) \in \theta^k \in \bar{e}^k$ and θ^k is a germ of $m_{E_M}(p)$. Now, we prove that $\bar{e}^1, \ldots, \bar{e}^m$ is a basis for $T^*_p(M, E_M)$.

Indeed, let $c_1, \ldots, c_m$ be reals such that $c_h\bar{e}^h = 0$. Then there exist germs $\lambda_1, \lambda'_1, \ldots, \lambda_s, \lambda'_s$ of $m_{E_M}(p)$ such that

$$c_h\theta^h = \lambda_1\lambda'_1 + \ldots + \lambda_s\lambda'_s.$$

Hence it follows that for some functions $\gamma_1, \gamma'_1, \ldots, \gamma_s, \gamma'_s$ in E_M vanishing at p we have

$$c_h(e^h|M)|V = (\gamma_1|V)(\gamma'_1|V) + \ldots + (\gamma_s|V)(\gamma'_s|V), \tag{4.2}$$

where V is some neighbourhood of p, open in the natural topology of the set M. Equalities (4.2) and (4.1) give

$$0 = \sum_{t=1}^{s} \left(e'_i(\gamma_t)\gamma'_t(p) + \gamma_t(p)e'_i(\gamma_t)\right) = c_h e'_i(e^h|M) = c_h\delta^h_i = c_i.$$

Then the system $\bar{e}^1, \ldots, \bar{e}^m$ is linearly independent. From Proposition 1.2 it follows that this system is a basis for $T^*_p(M, E_M)$.

Therefore, we may express the elements $\bar{e}^{m+1}, \ldots, \bar{e}^n$ linearly by $\bar{e}^1, \ldots, \bar{e}^m$. So there exist reals a^j_i ($i = 1, \ldots, m$ and $j = m+1, \ldots, n$) such that $\bar{e}^j = a^j_i\bar{e}^i$. Then there exist functions $a^j_1, \beta^j_1, \ldots, a^j_N, \beta^j_N$ in E and a neighbourhood U of p open in $\boldsymbol{R}^n$ such that for $j = m+1, \ldots, n$ we have

$$e^j|M \cap U = \left(a^j_i e^i + \sum_{\varrho=1}^{N} a^j_\varrho\beta^j_\varrho\right)^{\cdot} | M \cap U. \tag{4.3}$$

Applying the isometry $\boldsymbol{R}^n \ni (u^1, \ldots, u^n) \mapsto p + u^k e_k$ we may assume that $p = (0, \ldots, 0)$ and $e_k = (\delta^1_k, \ldots, \delta^n_k)$, $k = 1, \ldots, n$, as well as that the hyperplane M_p is of the form $\boldsymbol{R}^m \times 0$ and the orthogonal complement to M_p passing through p takes the form $0 \times \boldsymbol{R}^{n-m}$. Therefore, equality (4.3) is equivalent to

(4.4) $$u^j = F^j(u) \quad \text{for } u \in M \cap U,\ j = m+1, \ldots, n,$$

where for any $u = u^k e_k \in \boldsymbol{R}^n$ we set

(4.5) $$F^j(u) = a^j_i u^i + \sum_{\varrho=1}^{N} \alpha^j_\varrho(u)\beta^j_\varrho(u).$$

The functions F^j belong to E. Therefore, we have got the functions F^j_k of class C^∞ such that, for any $t = t^k e_k$ and $u = u^k e_k$ in $\boldsymbol{R}^n$,

(4.6) $$F^j(t) - F^j(u) = F^j_k(t, u)(t^k - u^k), \qquad F^j_k(u, u) = F^j_{|k}(u),$$

where $F^j_{|k}$ stands for the partial derivative of the function F^j with respect to the k-th variable. From (4.5) and the equalities $\alpha^j_\varrho(p) = 0 = \beta^j_\varrho(p)$ it follows that for $j' = m+1, \ldots, n$ we have

$$F^j_{|j'}(p) = \sum_{\varrho=1}^{N} \alpha^j_{\varrho|j'}(p)\beta^j_\varrho(p) + \alpha^j_\varrho(p)\beta^j_{\varrho|j'}(p) = 0.$$

Let $r > 0$ be a number such that $B(p, r) \subset U$, let p_{rM} be continuous with respect to its domain (Proposition 3.2), and

(4.7) $$\det[\delta^j_{j'} - F^j_{j'}(t, u);\ m+1 \leqslant j', j \leqslant n] \neq 0 \qquad \text{for } t, u \in B(p, r).$$

Denote the inverse matrix of $[\delta^j_{j'} - F^j_{j'}(t, u);\ m+1 \leqslant j', j \leqslant n]$ by

$$[G^{j'}_j(t, u);\ m+1 \leqslant j, j' \leqslant n].$$

Now, let us set $f = p_{rM}$, where p_{rM} has been defined at the beginning of this section. Conditions (i)-(iii) follow directly from the definition of the mapping p_{rM} and Propositions 2.2 and 3.2. It is easy to check that for every u of the domain of f we have

$$f(u) = u^1 e_1 + \ldots + u^m e_m + f^{m+1}(u) e_{m+1} + \ldots + f^n(u) e_n$$

or, shortly, $f(u) = u^i e_i + f^j(u) e_j$. From (4.4) we obtain

(4.8) $$f^j(u) = F^j\big(u^1, \ldots, u^m, f^{m+1}(u), \ldots, f^n(u)\big) \qquad \text{for } u \in D_f,$$

where D_f denotes the domain of f. Let $t, u \in D_f$. From (4.6) and (4.8) we get

$$f^j(t) - f^j(u) = F^j_i(\tilde{t}, \tilde{u})(t^i - u^i) + F^j_{j'}(\tilde{t}, \tilde{u})\big(f^{j'}(t) - f^{j'}(u)\big), \qquad j = m+1, \ldots, n,$$

where

(4.9) $$\begin{aligned} \tilde{t} &= \big(t^1, \ldots, t^m, f^{m+1}(t), \ldots, f^n(t)\big), \\ \tilde{u} &= \big(u^1, \ldots, u^m, f^{m+1}(u), \ldots, f^n(u)\big). \end{aligned}$$

This is equivalent to

$$\big(f^{j'}(t)-f^{j'}(u)\big)\big(\delta^j_{j'}-F^j_{j'}(\tilde{t},\tilde{u})\big) = F^j_i(\tilde{t},\tilde{u})(t^i-u^i),$$

whence

$$f^j(t)-f^j(u) = F^{j'}_i(\tilde{t},\tilde{u})G^j_{j'}(\tilde{t},\tilde{u})(t^i-u^i), \quad j = m+1, \ldots, n.$$

Setting $H^j_i(t,u) = F^{j'}_i(t,u)G^j_{j'}(t,u)$ for any $t, u \in B(p,r)$, we have the functions H^j_i of class C^∞ on $B(p,r)\times B(p,r)$ such that

$$f^j(t)-f^j(u) = H^j_i(\tilde{t},\tilde{u})(t^i-u^i) \quad \text{for } t, u \in D_f, \tag{4.10}$$

where $\tilde{t}$ and $\tilde{u}$ are given by (4.9).

Let $q \in D_f$ and let v be any direction of the set D_f at the point q. For any pair of sequences $t_1, t_2, \ldots$ and $u_1, u_2, \ldots$ of points in D_f such that $t_h \neq u_h$, $h = 1, 2, \ldots$, $t_h \to q$, $u_h \to q$, $(t_h-u_h)|t_h-u_h|^{-1} \to v$ as $h \to \infty$, from (4.10), because of the continuity of f, we obtain

$$\frac{f^j(t_h)-f^j(u_h)}{|t_h-u_h|} = H^j_i(\tilde{t}_h,\tilde{u}_h)\frac{t^i_h-u^i_h}{|t_h-u_h|} \to H^j_i(\tilde{q},\tilde{q})v^i \quad \text{as } h \to \infty,$$

where $\tilde{q} = \big(q^1, \ldots, q^m, f^{m+1}(q), \ldots, f^n(q)\big)$, $v = (v^1, \ldots, v^m, 0, \ldots, 0)$. The number $H^j_i(\tilde{q},\tilde{q})v^i$ does not depend on the choice of the sequences $t_1, t_2, \ldots$ and $u_1, u_2, \ldots$, is the derivative of the function f^j at the point q with respect to the direction v, and is denoted by $\partial_v f(q)$. Therefore, we have the equalities

$$\partial_v f^j(q) = H^j_i\big(q^1, \ldots, q^m, f^{m+1}(q), \ldots, f^n(q)\big)v^i, \quad j = m+1, \ldots, n,$$

where for abbreviation we write $H^j_i(t)$ instead of $H^j_i(t,t)$. Since H^j_i are of class C^∞ on $B(p,r)$, the derivatives of all orders exist at every point of the domain of f with respect to all directions of this domain, which completes the proof.

Let $f\colon X \to Y$ be a continuous mapping of the topological spaces. This mapping is said to be *locally open* at the point p in X if for any neighbourhood V of p there exists a neighbourhood U of p open in X, $U \subset V$, such that the image $f(U)$ is open in Y.

THEOREM 4.2. *For any subset M of $\boldsymbol{R}$ the following conditions are equivalent:*

(v) *M is a regular m-dimensional hypersurface of class C^∞ in $\boldsymbol{R}^n$;*

(vi) *for any $p \in M$ the orthogonal projection π_p of M, with topology induced from $\boldsymbol{R}^n$, onto the tangent hyperplane M_p to the differential space (M, E_M) is the locally open mapping at the point p.*

Proof. It is evident that (vi) follows from (v). To prove that (vi) implies (v), take any $p \in M$. According to Theorem 4.1 there exists a function f fulfilling conditions (i)-(iv). Then there exists a neighbourhood U of p open in M and contained in $B(p, r)$, where the set of values of f coincides with $M \cap B(p, r)$, and such that $\pi_p(U)$ is open in M_p. It is easy to verify that $\pi_p(U)$ is contained in the domain of f and $f(\pi_p(U)) = M \cap V_0$, where $V_0 = B(p, r) \cap \pi_p^{-1}(\pi_p(U))$. The set V_0 is open in $\boldsymbol{R}^n$. According to (iv) the mapping f is of class C^∞ on $\pi_p(U)$, and by (iii) it is regular on this set, which completes the proof.

Theorem 4.1 is very useful in finding the tangent hyperplane. From this theorem it follows, for example, that if φ is a real continuous function on $\boldsymbol{R}$ without the derivative at any point, then the tangent hyperplane M_p, where M is the graph of φ, coincides with $\boldsymbol{R}^2$ at any point of M, while the topological dimension of M is obviously equal to 1. Similarly, the number $\dim M_p - \operatorname{top}\dim_p M$, where $\operatorname{top}\dim_p M$ stands for the dimension of the set M at the point p regarded together with topology induced from the Cartesian space, is a positive integer in the case where M is the set of all $(x, y) \in \boldsymbol{R}^2$ such that $y(1+x^2\ln y) = 0$. Another example of the same type is given for

$$M = \{(x, 0); x \leqslant 0\} \cup \{(x, x^2); x > 0\}, \quad p = (0, 0).$$

It seems remarkable that for this set the hyperplane M_p^d of all directions is 1-dimensional. The number $\dim M_p - \dim M_p^d$ can also be useful in characterizing the singularity of a subset of $\boldsymbol{R}^n$.

REFERENCES

[1] S. Mac Lane, *Differentiable spaces*, p. 1-9 in: Notes for Geometrical Mechanics, Winter 1970 (unpublished).

[2] M. Pustelnik, *On some isomorphisms of the generalized tangent structures*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 19 (1971), p. 67-72.

[3] J. P. Serre, *Lie algebras and Lie groups*, New York - Amsterdam - Benjamin 1965.

[4] R. Sikorski, *Abstract covariant derivative*, Colloquium Mathematicum 18 (1967), p. 251-272.

[5] — *Introduction to differential geometry* (in Polish), Warszawa 1972.

[6] W. Waliszewski, *Regular and coregular mappings of differential spaces*, Annales Polonici Mathematici 30 (1975), p. 263-281.

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