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CONTRACTORS AND FIXED POINTS

BY

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1. Introduction. With a view to providing a unified approach to solving general equations in abstract spaces by iterative methods, Altman [1] introduced the theory of contractors (cf. also [2] and its references). For example, in this manner the Banach contraction principle (BCP) and Krasnosel'skiĭ's fixed point theorem have been unified. Further recent results of Reddy and Subrahmanyam [14]–[16] unify Altman's contractor theorem (Theorem 2.3 below) and fixed point theorems of Matkowski [10], Krasnosel'skiĭ [8] and Czerwik [5]. However, some nice generalizations of the BCP and Nadler's multivalued contraction principle [11] (e.g., Theorems 2.1 and 2.2 below) cannot be obtained from [1], [2] or [15]–[17].

Herein, we prove, in Section 3, a general contractor theorem which includes Altman's Theorem 2.3 and several fixed point theorems and other results for contractive type single- and multivalued operators.

The Mann iteration scheme to approximate solutions of operator equations and fixed points of contractive operators has been widely studied (see, e.g., [3], [6], [9], [13], [19]). In Section 4 an iteration scheme is introduced which generalizes the Mann iteration and we show that if it converges, then it converges to a solution (Theorem 4.1).

2. Contractors. Consistent with [12], p. 620, we will use the following notation where Y is a Banach space:

$$\mathrm{CL}(Y) = \{A \subseteq Y\colon\ A \neq \emptyset \text{ and is closed}\}.$$

For A, $B \in \mathrm{CL}(Y)$ and $\varepsilon > 0$,

$$N(\varepsilon, A) = \{y \in Y\colon\ \|y-a\| < \varepsilon \text{ for some } a \in A\},$$

$$E_{A,B} = \{\varepsilon > 0\colon\ A \subseteq N(\varepsilon, B),\ B \subseteq N(\varepsilon, A)\},$$

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset, \\ +\infty & \text{if } E_{A,B} = \emptyset, \end{cases}$$

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and for $y \in Y$

$$D(y, A) = \inf\{\|y-a\|: a \in A\}.$$

H is called the *generalized Hausdorff metric* for $CL(Y)$.

The following is a Banach space version of Pal and Maiti's result [14] (see also [19], p. 42):

THEOREM 2.1. *Let T be an operator on a Banach space Y such that, for any two elements $x, y \in Y$, at least one of the following is true*:

$$\|x-Tx\|+\|y-Ty\| \leqslant a\|x-y\|, \quad 1 < a < 2; \tag{2.1}$$

$$\|x-Tx\|+\|y-Ty\| \leqslant b(\|x-Ty\|+\|y-Tx\|+\|x-y\|), \quad 1/2 < b < 2/3: \tag{2.2}$$

$$\|Tx-Ty\| \leqslant k\max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Ty\|+\|y-Tx\|}{2}\right\}, \quad 0 < k < 1; \tag{2.3}$$

$$\|x-Tx\|+\|y-Ty\|+\|Tx-Ty\| \leqslant c(\|x-Tx\|+\|y-Ty\|), \quad 1 < c < 3/2. \tag{2.4}$$

Then T has a fixed point.

THEOREM 2.2 (Ćirić [4]). *Let Y be a Banach space and $F: Y \to CL(Y)$ satisfy*

$$H(Fx, Fy) \leqslant k\max\left\{\|x-y\|, D(x, Fx), D(y, Fy), \frac{D(x, Fy)+D(y, Fx)}{2}\right\}$$

for all $x, y \in Y$ and some $k \in (0, 1)$. Then F has a fixed point.

These theorems are proved for orbitally complete metric spaces (see [4] and [14]).

DEFINITION 2.1 (Altman [1]). Let X and Y be Banach spaces, $P: D(P) \subset X \to Y$ be a nonlinear map with domain $D(P)$ and $\Gamma(x): Y \to X$ be a bounded linear operator associated with $x \in X$. The map P is said to have a *contractor* $\Gamma(x)$ if there is k $(0 < k < 1)$ such that

$$x+\Gamma(x)y \in D(P) \quad \text{for } x \in D(P),\ y \in Y; \tag{2.5}$$

$$\|P(x+\Gamma(x)y)-Px-y\| \leqslant k\|y\| \quad \text{for } x \in D(P),\ y \in Y. \tag{2.6}$$

A contractor $\Gamma(x)$ is said to be *regular* if (2.6) is satisfied for all $y \in Y$ and $D(P) = \Gamma(x)Y$. The operator P is said to be *closed* on $D(P)$ if the graph of P is closed, i.e., if $x_n \in D(P)$, $x_n \to x$ and $Px_n \to y$, then $x \in D(P)$ and $y = Px$. In the case of a nonlinear multivalued operator $P: D(P) \to CL(Y)$, P is *closed* on $D(P)$ if $x_n \to x$, $y_n \in Px_n$ and $y_n \to y$ imply $x \in D(P)$ and $y \in Px$ (see [17]).

The following existence theorem, due to Altman, is fundamental to the theory of contractors.

THEOREM 2.3 ([1], p. 13). *Suppose that the closed nonlinear operator* P: $D(P) \subseteq X \to Y$ *has a bounded contractor* Γ *satisfying* (2.5), (2.6) *and*

$$\|\Gamma(x)\| \leqslant B \quad \text{for all } x \in D(P).$$

Then the equation $Px = y$ *has a solution for each* $y \in Y$. *When* Γ *is regular,* (2.5) *always holds and the solution is unique.*

3. Multivalued mappings and general contractors. The following lemma (cf. [20]) will be used:

LEMMA 3.1. *Let* A, $B \in \mathrm{CL}(Y)$ *and* $a \in A$. *Then for* $k \in (0, 1)$ *and* $\lambda \in [0, 1)$ *there exists* $b \in B$ *such that*

$$\|a-b\| \leqslant k^{-\lambda} H(A, B).$$

(Note that for $k \in (0, 1)$ it is always possible to choose $\lambda \in [0, 1)$ such that $1 \leqslant k^{-\lambda} \leqslant 2$.)

Let X and Y be Banach spaces, P: $D(P) \subseteq X \to \mathrm{CL}(Y)$ and $\Gamma(x)$: $Y \to X$ be a bounded linear operator. For convenience, define $t_i = t_i(x, y)$, $i = 1, \ldots, 5$, for $x \in D(P)$, $y \in Y$ as follows:

$$t_1 = \big(P(x+\Gamma(x)y),\ y+Px\big), \qquad t_2 = (y,\ y-Px),$$
$$t_3 = \big(x,\ x-\Gamma(x)\big(P(x+\Gamma(x)y)\big)\big), \qquad t_4 = (y,\ -Px),$$
$$t_5 = \big(x,\ x-\Gamma(x)\big(-y+P(x+\Gamma(x)y)\big)\big).$$

THEOREM 3.1. *Suppose* P: $D(P) \subseteq X \to \mathrm{CL}(Y)$ *satisfies the following: there exists a bounded linear operator* $\Gamma(x)$: $Y \to X$ *associated with* $x \in X$ *such that*

(3.1) $$\|\Gamma(x)\| \leqslant B, \quad B > 0,\ x \in D(P);$$

(3.2) $$x+\Gamma(x)y \in D(P) \quad \text{whenever } x \in D(P),\ y \in Y.$$

Further, for $x \in D(P)$, $y \in Y$, *at least one of the following holds*:

(3.3) $$Ht_2+Ht_3 \leqslant a\|y\|, \quad 1 < a < 1+B;$$

(3.4) $$Ht_2+Ht_3 \leqslant b\,[Dt_4+Dt_5+\|y\|], \quad (1+B)^{-1} < b < 1-B(1+2B)^{-1};$$

(3.5) $$Ht_1 \leqslant k \max\left\{\|y\|,\ Dt_2,\ Dt_3,\ \frac{Dt_4+Dt_5}{2}\right\},$$

$$0 < k < 1,\ Bk^{1-\lambda} < 1, \text{ where } \lambda \in [0, 1) \text{ is such that } k^{-\lambda} \leqslant 2;$$

(3.6) $$Ht_1+Ht_2+Ht_3 \leqslant c\,[Dt_4+Dt_5], \quad 1/B < c < 1+B/2.$$

Then there exists $z \in D(P)$ *such that* $0 \in Pz$.

Proof. Construct two sequences $\{x_n\} \subseteq D(P)$ and $\{y_n\} \subseteq Y$ as follows: Choose $x_0 \in D(P)$ and $y_0 \in Px_0$. Set

$$x_1 = x_0 - \Gamma(x_0) y_0 \in D(P)$$

and note that $0 \in Px_0 - y_0$. By Lemma 3.1, choose $y_1 \in Px_1$ such that

$$\|y_1\| \leqslant k^{-\lambda} H(Px_1, Px_0 - y_0).$$

Set $x_2 = x_1 - \Gamma(x_1) y_1$ and choose $y_2 \in Px_2$ such that

$$\|y_2\| \leqslant k^{-\lambda} H(Px_2, Px_1 - y_1).$$

Having chosen x_n and $y_n \in Px_n$, let $x_{n+1} = x_n - \Gamma(x_n) y_n$ and choose $y_{n+1} \in Px_{n+1}$ such that

$$\|y_{n+1}\| \leqslant k^{-\lambda} H(Px_{n+1}, Px_n - y_n). \tag{3.7}$$

Now for $x = x_n$ and $y = -y_n$ we will consider each of the cases (3.3)–(3.6). Frequent use will be made of the fact that, for any $u \in D(P)$ and any $v \in A \in \mathrm{CL}(Y)$,

$$\|u - v\| \leqslant H(u, A).$$

In case (3.3) we obtain

$$H(-y_n, -y_n - Px_n) + H(x_n, x_n - \Gamma(x_n) Px_{n+1}) \leqslant a \|y_n\|,$$

that is, $\|y_n\| + B\|y_{n+1}\| \leqslant a\|y_n\|$. Therefore

$$\|y_{n+1}\| \leqslant q_1 \|y_n\|, \quad \text{where } q_1 = \frac{a-1}{B}. \tag{3.3a}$$

In case (3.4) we have

$$\|y_n\| + B\|y_{n+1}\| \leqslant b[\|y_n + y_{n+1}\| + \|y_n\|].$$

Thus

$$\|y_{n+1}\| \leqslant q_2 \|y_n\|, \quad \text{where } q_2 = \frac{b(1+B)-1}{B(1-b)}. \tag{3.4a}$$

In case (3.5) we obtain

$$H(Px_{n+1}, -y_n + Px_n) \leqslant k \max\Big\{\|y_n\|, D(-y_n, y_n - Px_n),$$
$$D(x_n, x_n - \Gamma(x_n) Px_{n+1}), \frac{D(-y_n, -Px_n) + D(x_n, x_n - \Gamma(x_n)(y + Px_{n+1}))}{2}\Big\}$$
$$\leqslant k \max\Big\{\|y_n\|, B\|y_{n+1}\|, \frac{B}{2}\|y_n + y_{n+1}\|\Big\}.$$

Using (3.7), this yields

$$\|y_{n+1}\| \leqslant q_3 \|y_n\|, \quad \text{where } q_3 = \max\left\{k^{1-\lambda}, \frac{Bk^{1-\lambda}}{2-Bk^{1-\lambda}}\right\}. \tag{3.5a}$$

Finally, in case (3.6) we have

$$\begin{aligned} k^{1-\lambda}(\|y_n\| + B\|y_{n+1}\|) + \|y_{n+1}\| &\leqslant k^{1-\lambda}[Ht_1(x_n, -y_n) + Ht_2(x_n, -y_n) + Ht_3(x_n, -y_n)] \\ &\leqslant k^{1-\lambda} c[0 + B\| y_n + y_{n+1}\|], \end{aligned}$$

whence

$$\|y_{n+1}\| \leqslant q_4 \|y_n\|, \quad \text{where } q_4 = \frac{Bc-1}{k^{1-\lambda} - Bc + B}. \tag{3.6a}$$

By (3.3a)–(3.6a), $\|y_{n+1}\| \leqslant q\|y_n\|$ for all n, where $q = \max\{q_1, q_2, q_3, q_4\} < 1$. Hence, since $x_{n+1} = x_n - \Gamma(x_n) y_n$, $\{x_n\}$ is a Cauchy sequence and $x_n \to z$ and $y_n \to 0$. Consequently, $0 \in Pz$, since P is closed.

Several contractor theorems follow as corollaries to Theorem 3.1.

COROLLARY 3.1. *If P and $\Gamma(x)$ are as above and satisfy* (3.1), (3.2) *and*

$$Ht_1 \leqslant k \max\left\{\|y\|, Dt_2, Dt_3, \frac{Dt_4 + Dt_5}{2}\right\} \tag{3.8}$$

for $x \in D(P)$, $y \in Y$ and some $k \in (0, 1)$ such that $Bk^{1-\lambda} < 1$ for $\lambda \in [0, 1)$, then there exists $z \in D(P)$ such that $0 \in Pz$.

Note that in Theorem 3.1 and Corollary 3.1, if all members of $\mathrm{CL}(Y)$ are compact, then one may choose $\lambda = 0$. Moreover, the conclusion of Corollary 3.1 remains true if (3.8) is replaced by

$$Ht_1 \leqslant \alpha\|y\| + \beta Dt_2 + \gamma Dt_3 + \delta(Dt_4 + Dt_5) \tag{3.9}$$

for $x \in D(P)$, $y \in Y$ and nonnegative numbers α, β, γ, δ with $0 < k = \alpha + \beta + \gamma + 2\delta < 1$. In fact, we obtain the following

COROLLARY 3.2. *If P and $\Gamma(x)$ are as above and satisfy* (3.1), (3.2) *and* (3.9) *for $x \in D(P)$, $y \in Y$ and $\alpha, \beta, \gamma, \delta \geqslant 0$ such that*

$$k = \alpha + \beta + \gamma + 2\delta < 1 \quad \text{and} \quad (\alpha + \beta + B(\gamma + 2\delta))k^{-\lambda} < 1$$

for some $\lambda \in [0, 1)$, then there exists $z \in D(P)$ such that $0 \in Pz$.

Corollary 3.2 with $\beta = \gamma = \delta = 0$ yields a multivalued version of Altman's Theorem 2.3. Also, the main result of Reddy and Subrahmanyam ([17], Theorem 3.1) with $n = 1$ is obtained by setting $\delta = 0$.

If the hypotheses (3.1)–(3.6) of Theorem 3.1 are satisfied, we may say that $\Gamma(x)$ is a *general contractor* for P and that Theorem 3.1 is a *multivalued* *contractor theorem*. Similarly, for P and $\Gamma(x)$ satisfying the hypotheses of the next corollary, $\Gamma(x)$ may be called a *general contractor* of the single-valued mapping P and the result a *general contractor analog* of Theorem 3.1.

COROLLARY 3.3. *If $P\colon D(P) \subseteq X \to Y$ satisfies* (3.1), (3.2) *and, for $x \in D(P)$, $y \in Y$, at least one of the following*

$$(3.10) \qquad \|Px\| + \|\Gamma(x)(P(x+\Gamma(x)y))\| \leqslant a\|y\|, \qquad 1 < a < 1+B;$$

$$(3.11) \qquad \|Px\| + \|\Gamma(x)(P(x+\Gamma(x)y))\| \leqslant b\left[\|y+Px\| + \|\Gamma(x)(-y+P(x+\Gamma(x)y))\| + \|y\|\right], \quad (1+B)^{-1} < b < 1 - B(1+2B)^{-1};$$

$$(3.12) \qquad \|t_1\| \leqslant k \max\left\{\|y\|, \|Px\|, \|\Gamma(x)P(x+\Gamma(x)y)\|, \frac{\|y+Px\| + \|\Gamma(x)(-y+P(x+\Gamma(x)y))\|}{2}\right\}, \quad 0 < k < 1,\ kB < 1,$$

where $\|t_1\| = \|P(x+\Gamma(x)y) - y - Px\|$;

$$(3.13) \qquad \|Px\| + \|\Gamma(x)(P(x+\Gamma(x)y))\| + \|t_1\| \leqslant c\left[\|y+Px\| + \|\Gamma(x)(-y+P(x+\Gamma(x)y))\|\right], \quad 1/B < c < 1+B/2.$$

then $Px = 0$ has a solution in $D(P)$.

Proof. (3.3)–(3.6) reduce to (3.10)–(3.13) in the single-valued case.

COROLLARY 3.4. *Suppose $P\colon D(P) \subseteq X \to Y$ is closed and satisfies* (3.1), (3.2) *and* (3.12). *Then $Px = 0$ has a solution in $D(P)$. Further, the solution is unique when Γ is regular and $B \leqslant 1$.*

Proof. A solution exists, so we need to show the uniqueness. Let Γ be regular and α, β be two solutions of $Px = 0$. Then $\beta = \alpha + \Gamma(\alpha)y$ for some $y \in Y$. It suffices to show that $y = 0$. If $y \neq 0$, then

$$\|\beta - \alpha\| \leqslant B\|y\|$$

and

$$\|y\| = \|P\beta - P\alpha - y\| = \|P(\alpha + \Gamma(\alpha)y) - P\alpha - y\| \leqslant k \max\left\{\|y\|, 0, 0, \frac{\|y\| + \|\Gamma(\alpha)(-y)\|}{2}\right\}.$$

Thus $\|y\| \leqslant k\|y\|$ and $y = 0$.

Altman's Theorem 2.3 is included in Corollary 3.4. In fact, replacing (3.12) by

$$\|t_1\| \leqslant k \max\{\|y\|, \|Px\|, \|\Gamma(x)P(x+\Gamma(x)y)\|\},$$

we see by the above proof that it is not necessary that $B \leqslant 1$. Further, if P is single-valued and Γ is regular in Corollary 3.2, then the solution is unique provided $\delta(1+B) < 1-\alpha$.

Now a fixed point theorem is derived from Theorem 3.1 which unifies several fixed point theorems including Theorems 2.1 and 2.2.

COROLLARY 3.5. *Let $F\colon Y \to \mathrm{CL}(Y)$ satisfy, for $x, y \in Y$, at least one of the following*:

$$H(x, Fx)+H(y, Fy) \leqslant a\|x-y\|, \quad 1 < a < 2;$$

$$H(x, Fx)+H(y, Fy) \leqslant b[D(x, Fy)+D(y, Fx)+\|x-y\|], \quad 1/2 < b < 2/3;$$

$$H(Fx, Fy) \leqslant k \max\left\{\|x-y\|, D(x, Fx), D(y, Fy), \frac{D(x, Fy)+D(y, Fx)}{2}\right\}, \quad 0 < k < 1;$$

$$H(x, Fx)+H(y, Fy)+H(Fx, Fy) \leqslant c[D(x, Fy)+D(y, Fx)], \quad 1 < c < 3/2.$$

Then F has a fixed point.

Proof. Setting $\Gamma(x) = I$, the identity on $X = Y$, and $Px = x-Fx$ in Theorem 3.1, it is seen that $\{x_n\}$ converges to a fixed point of F.

Clearly, Theorem 2.2 is included in the above corollary, and setting $Fx = \{Tx\}$, for $T\colon Y \to Y$ we obtain Theorem 2.1. Further, setting $\Gamma(x) = I$, $X = Y$ and $Px = x-Fx$ in Corollary 3.1, a variant of fixed point theorems for multivalued operators due to Nadler [11] and Iseki [7] is obtained.

4. Approximation of solutions. In the single-valued general contractor theorem (Corollary 3.3) $\{x_n\}$, defined by $x_{n+1} = x_n-\Gamma(x_n)Px_n$, converges to a solution. This scheme was used by Altman in Theorem 2.3. We now introduce another scheme for a multivalued operator $P\colon D(P) \subseteq X \to \mathrm{CL}(Y)$ (resp. single-valued operator $P\colon D(P) \subseteq X \to Y$), as follows:

(4.1) $$x_0 \in X;$$

(4.2) $$x_{n+1} = x_n-\alpha_n\Gamma(x_n)y_n, \quad y_n \in P(x_n) \text{ (resp., } y_n = Px_n);$$

(4.3) $$0 \leqslant \alpha_n \leqslant 1, \quad \lim\alpha_n = \alpha > 0.$$

We denote the sequence $\{x_n\}$ defined above by $M(\Gamma(x_0), \alpha_n, P)$.

For $X = Y$, $T\colon Y \to Y$, defining $Px = x-Tx$ and $\Gamma(x) = I$, we see that (4.2) is equivalent to

(4.2′) $$x_{n+1} = (1-\alpha_n)x_n+\alpha_n Tx_n,$$

which is the Mann iterative process (see, e.g., [6], [18], [19]). A sequence $\{x_n\}$ satisfying (4.1), (4.2′) and (4.3) will be denoted by $M(x_0, \alpha_n, T)$.

Recently, Kuhfittig [9] has studied the Mann iterative process for certain classes of multivalued operators; in fact, point-compact operators. In Corollary 4.1 we will consider point-closed operators satisfying a very general contractive type condition which may include operators studied in [9].

THEOREM 4.1. *Under the hypotheses of Theorem* 3.1, $M(\Gamma(x_0), \alpha_n, P)$ *converges to a solution provided it converges,* $\Gamma(x)$: $Y \to X$ *is invertible and* $\Gamma(x)^{-1}$ *is continuous.*

Proof. Theorem 3.1 guarantees that P has a solution. Assume that the sequence $M(\Gamma(x_0), \alpha_n, P)$ converges to z. Then, since $\lim \alpha_n = \alpha > 0$ and $\Gamma(x_n)$ has a bounded inverse, the equality

$$\|x_{n+1} - x_n\| = \alpha_n \|\Gamma(x_n) y_n\|$$

implies $y_n \to 0$. Thus $0 \in Pz$, since P is closed.

Evidently, not all the hypotheses are needed in the proof of Theorem 4.1. In fact, if P is any closed operator (single-valued or multivalued) on $D(P)$ and if $M(\Gamma(x_0), \alpha_n, P)$ converges to z, then $0 \in Pz$ provided $\Gamma(x)$ has a bounded inverse. This suggests that, while considering a special case of Theorem 4.1, one needs to require only those conditions which ensure that P is closed. Therefore we have the following:

Let $X = Y$ be a normed space and $C \subseteq X$ be closed and convex and F: $C \to \mathrm{CL}(C)$. Defining $Px = x - Fx$ and $\Gamma(x) = I$ and replacing (4.2) by

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n p_n, \quad p_n \in F x_n, \tag{4.2''}$$

a sequence defined by (4.1), (4.2″) and (4.3) will be denoted by $M(p_0, \alpha_n, F)$.

COROLLARY 4.1. *Suppose that* $M(p_0, \alpha_n, F)$ *converges to* $z \in C$. *If, for* $x, y \in M(p_0, \alpha_n, F) \cup \{z\}$, *one of the following holds:*

$$H(x, Fx) + H(y, Fy) \leqslant a\|x-y\|, \quad 1 < a < 2; \tag{4.4}$$

$$H(x, Fx) + H(y, Fy) \leqslant b[D(x, Fy) + D(y, Fx) + \|x-y\|], \quad 1/2 < b < 2/3; \tag{4.5}$$

$$H(Fx, Fy) \leqslant k \max\{t\|x-y\|, D(x, Fx) + D(y, Fy), D(x, Fy) + D(y, Fx)\}, \quad 0 < k < 1,\ t > 0; \tag{4.6}$$

$$H(x, Fx) + H(y, Fy) + H(Fx, Fy) \leqslant c[D(x, Fy) + D(y, Fx)], \quad 1 < c < 3/2, \tag{4.7}$$

then z *is a fixed point of* F.

Proof. Since $\{\alpha_n\}$ is bounded away from zero and $x_n \to z$, it follows from (4.2″) that

$$\|x_n - p_n\| \to 0 \quad \text{and} \quad \|p_n - z\| \to 0.$$

Now suppose that (4.4) holds for the pair $x = x_n$, $y = z$. Then

$$\text{(4.4a)} \qquad \|x_n - p_n\| + D(z, Fz) \leqslant H(x_n, Fx_n) + H(z, Fz) \leqslant a\|x_n - z\|.$$

Similarly, if (4.5)–(4.7) are true, then correspondingly we obtain

$$\text{(4.5a)} \qquad \|x_n - p_n\| + D(z, Fz) \leqslant b[\|x_n - z\| + D(z, Fz) + \|z - p_n\| + \|x_n - z\|],$$

$$\begin{aligned} \text{(4.6a)} \qquad D(z, Fz) &\leqslant \|z - p_n\| + H(Fx_n, Fz) \\ &\leqslant \|z - p_n\| + k\max\{t\|x_n - z\|, D(x_n, Fx_n) + D(z, Fz), \\ &\qquad\qquad D(x_n, Fz) + D(z, Fx_n)\} \\ &\leqslant \|z - p_n\| + k\max\{t\|x_n - z\|, \|x_n - p_n\| + D(z, Fz), \\ &\qquad\qquad 2\|x_n - z\| + D(z, Fz) + \|z - p_n\|\}, \end{aligned}$$

$$\begin{aligned} \text{(4.7a)} \qquad & \|x_n - p_n\| + 2D(z, Fz) \\ &\leqslant H(x_n, Fx_n) + H(z, Fz) + H(z, Fz) + H(Fx_n, Fz) + \|z - p_n\| \\ &\leqslant c[D(x_n, Fz) + D(z, Fx_n)] + \|z - p_n\| \\ &\leqslant c[\|x_n - z\| + D(z, Fz) + \|z - p_n\|] + \|z - p_n\|. \end{aligned}$$

Letting $n \to \infty$ in (4.4a)–(4.7a) we obtain $D(z, fz) = 0$, so $z \in Fz$.

Finally, the following is a variant of a result of Rhoades ([19], Theorems 1 and 2):

COROLLARY 4.2. *Let C be a closed, convex subset of a normed space, $T\colon C \to C$ and $M(x_0, \alpha_n, T)$ converge to $z \in C$. If, for $x, y \in M(x_0, \alpha_n, T) \cup \{z\}$, one of* (2.1), (2.2), (2.4) *and*

$$\|Tx - Ty\| \leqslant k\max\{t\|x - y\|, \|x - Tx\| + \|y - Ty\|, \|x - Ty\| + \|y - Tx\|\}, \qquad 0 < k < 1,\ t > 0,$$

holds, then z is a fixed point of T.

Proof. Write $Fx = \{Tx\}$, $x \in C$. Then

$$M(x_0, \alpha_n, T) = M(p_0, \alpha_n, F)$$

and the result follows from Corollary 4.1.

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