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REMARKS ON NON-PLANABLE DENDROIDS

BY

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1. The paper concerns the problem of imbeddability of some curves (i.e., one-dimensional continua) in the Euclidean plane. We recall here some earlier results and examples related to planability which are either well-known theorems or — sometimes — some popular facts in this area, that we need in our further applications to state some conditions of non-planability of dendroids.

All spaces considered in this paper are metric. The Euclidean plane is denoted by R^2. A subset A of a space X is said to be *planable* if it is imbeddable in the plane, i.e., if there is a homeomorphism $h\colon A \to h(A) \subset R^2$. It is easy to observe the following

PROPOSITION 1. *The property of being planable is hereditary, i.e., every subset of a planable space is planable.*

In other words, we have

PROPOSITION 1′. *If a space X contains a non-planable subspace Y, then X is non-planable.*

Some statements which concern non-planability of some spaces can be formulated by the use of the notion of accessibility. A point p is said to be *accessible* from a set D provided that there exists an arc xp lying in $D \cup \{p\}$ (see [20], p. 111). The notion of accessibility is not a topological invariant, i.e., the accessibility of a point $p \in X \subset R^2$ from $R^2 \setminus X$ depends on the manner of lying of X in R^2. It can be seen from the following example.

Let X be the cone over the set

$$\{(0,0)\} \cup \left\{\left(\frac{1}{2n}, 0\right)\colon n = 1, 2, \ldots\right\} \cup \left\{\left(\frac{-1}{2n-1}, 0\right)\colon n = 1, 2, \ldots\right\}$$

with the vertex at $(0,1)$. Thus X is the so-called *harmonic fan* having $\{(0, y)\colon 0 \leqslant y \leqslant 1\}$ as its limit segment. The point $p = (0, 1/2)$ is, therefore, inaccessible from the set $R^2 \setminus X$. But, if we consider the homeomorphism $h\colon X \to h(X) \subset R^2$ defined by

$$h(x, y) = \begin{cases} (x, y) & \text{if } x \geqslant 0, \\ (-x, y) & \text{if } x < 0, \end{cases}$$

then we get the cone over the set $\{(0,0)\} \cup \{(1/n, 0)\colon n = 1, 2, \ldots\}$ with the vertex at $(0,1)$ as $h(X)$, and we see that $p = h(p)$ is now accessible from the set $R^2 \setminus h(X)$.

Therefore, the following concept of the strong inaccessibility of points in subsets in the Euclidean plane seems to be useful for investigations of non-planable spaces.

Definition 1. A point $p \in S \subset R^2$ is said to be a *strongly inaccessible point* of S provided that there is no homeomorphism

$$h\colon S \to h(S) \subset R^2$$

such that $h(p)$ is accessible from the complement $R^2 \setminus h(S)$ (see [5]).

For example, if G is the closure of the graph in polar coordinates (ϱ, ϑ) of $\varrho = 1/\vartheta + |\sin(3\vartheta/2)|^{\vartheta}$, where $\vartheta \in [\pi, \infty)$, it is the union of a simple triod and a curve spiralling down on it. Each point of the triod is a strongly inaccessible point of G (cf. [1], Example 1, p. 654).

PROPOSITION 2. *Let a space X contain a planable subset S and an arc pq such that $pq \cap S = \{p\}$. If p is a strongly inaccessible point of S, then X is non-planable.*

Indeed, if X is planable, then there is a homeomorphism

$$h\colon X \to h(X) \subset R^2,$$

and $pq \cap S = \{p\}$ implies $h(pq) \cap h(S) = \{h(p)\}$. Thus $h(p)$ is an accessible point of $h(S)$ from $R^2 \setminus h(S)$.

In the sequel, let δ denote the Euclidean metric in the plane.

LEMMA 1. *If p is not a strongly inaccessible point of a planable continuum X, then there exists a homeomorphism $h\colon X \to h(X) \subset R^2$ such that $h(p)$ is accessible from the unbounded component of $R^2 \setminus h(X)$.*

Proof. Since p is not a strongly inaccessible point of X, thus, according to the definition, there is a homeomorphism

$$\varphi\colon X \to \varphi(X) \subset R^2$$

such that the point $\varphi(p)$ is accessible from $R^2 \setminus \varphi(X)$. Therefore, there exists a point $a \in R^2 \setminus \varphi(X)$ with the property that the points $\varphi(p)$ and a can be joined by an arc $\varphi(p)a$ such that

$$\big(\varphi(p)a \setminus \{\varphi(p)\}\big) \cap \varphi(X) = \emptyset.$$

If a belongs to the unbounded component of $R^2\setminus\varphi(X)$, then we put $h = \varphi$ and the proof is completed. In the opposite case, let

$$Q(a, \varepsilon) = \{x \in R^2\colon\ \delta(a, x) < \varepsilon\},$$

where $\varepsilon > 0$ is such that $\overline{Q(a, \varepsilon)} \subset R^2\setminus\varphi(X)$. Further, let q be the first point of the arc $\varphi(p)a$, ordered from $\varphi(p)$ to a, lying in the boundary $C = \overline{Q(a, \varepsilon)}\setminus Q(a, \varepsilon)$ of $Q(a, \varepsilon)$, i.e.,

$$q \in C \quad \text{and} \quad \varphi(p)q \subset R^2\setminus Q(a, \varepsilon).$$

Consider the inversion ψ of the plane R^2 with respect to the circle C (see, e.g., [9], p. 141), i.e.,

$$\psi\colon\ R^2\setminus\{a\} \to R^2\setminus\{a\}.$$

According to the well-known properties of the inversion we have

$$\psi\big(R^2\setminus\overline{Q(a, \varepsilon)}\big) = Q(a, \varepsilon)\setminus\{a\}$$

and $\psi|C$ is the identity. Since

$$\varphi(X) \subset R^2\setminus\overline{Q(a, \varepsilon)},$$

we have $\psi\big(\varphi(X)\big) \subset Q(a, \varepsilon)$ and, therefore,

$$\psi(q) = q \in C \subset R^2\setminus Q(a, \varepsilon) \subset R^2\setminus\psi\big(\varphi(X)\big).$$

Observe that $R^2\setminus Q(a, \varepsilon)$, as a connected and unbounded set, must be contained in the unbounded component K of $R^2\setminus\psi\big(\varphi(X)\big)$. Further, the inversion ψ being a one-to-one continuous mapping, the image of the arc $\varphi(p)q$ under ψ is an arc $A = \psi\big(\varphi(p)q\big)$ joining the points $\psi\big(\varphi(p)\big)$ and $\psi(q)$ and such that $A\setminus\psi\big(\varphi(p)\big) \subset K$. Thus we have

$$A \cap \psi\big(\varphi(X)\big) = \big\{\psi\big(\varphi(p)\big)\big\},$$

and to complete the proof it is enough to put $h = \psi|\varphi(X)$.

PROPOSITION 3. *Let a continuum X be the union of two non-degenerate planable continua X_1 and X_2 the intersection of which is a point p and such that there are non-degenerate arcs $pq_1 \subset X_1$ and $pq_2 \subset X_2$. Then the continuum X is non-planable if and only if p is a strongly inaccessible point of either X_1 or X_2.*

Proof. Assume that p is a strongly inaccessible point of X_1 and let pq_2 be an arc in X_2. Thus X is non-planable by Proposition 2.

Assume now that the continuum X is non-planable and suppose that p is not a strongly inaccessible point of both X_1 and X_2. Thus, by Lemma 1, there exist homeomorphisms h_1 and h_2,

$$h_1\colon\ X_1 \to h_1(X_1) \subset R^2 \quad \text{and} \quad h_2\colon\ X_2 \to h_2(X_2) \subset R^2,$$

for which the points $h_1(p)$ and $h_2(p)$ are accessible from the unbounded components of $R^2 \setminus h_1(X_1)$ and $R^2 \setminus h_2(X_2)$, respectively. We may assume, without loss of generality, that the images $h_1(X_1)$ and $h_2(X_2)$ of continua X_1 and X_2, respectively, are disjoint and, moreover, that $h_1(X_1)$ is contained in the unbounded component of $R^2 \setminus h_2(X_2)$ and, similarly, that $h_2(X_2)$ lies in the unbounded component of $R^2 \setminus h_1(X_1)$. Therefore, there is an arc in the plane R^2 which joins $h_1(p)$ with $h_2(p)$ and which lies, beside of its end points, out of the union $h_1(X_1) \cup h_2(X_2)$:

$$h_1(p)h_2(p) \setminus \{h_1(p), h_2(p)\} \subset R^2 \setminus \big(h_1(X_1) \cup h_2(X_2)\big).$$

Consider the union

$$h_1(X_1) \cup h_1(p)h_2(p) \cup h_2(X_2) \subset R^2$$

and the mapping $g\colon R^2 \to g(R^2)$ which shrinks the arc $h_1(p)h_2(p)$ to a point $g\big(h_1(p)\big) = g\big(h_2(p)\big)$ and which is a homeomorphism out of this arc. It follows that $g(R^2)$ is homeomorphic to the plane ([17], Theorem 22, p. 425; cf. [14], § 61, IV, Theorem 8, p. 533). Then the mapping

$$f\colon X \to f(X) \subset g(R^2)$$

defined by

$$f(x) = \begin{cases} g\big(h_1(x)\big) & \text{if } x \in X_1, \\ g\big(h_2(x)\big) & \text{if } x \in X_2 \end{cases}$$

is a homeomorphism imbedding X into the plane.

As an immediate consequence of Propositions 3 and 1′ we get

COROLLARY 1. *If a continuum X contains the union of two non-degenerate planable continua X_1 and X_2 the intersection of which is a point p that is a strongly inaccessible point of either X_1 or X_2 and such that there are non-degenerate arcs $pq_1 \subset X_1$ and $pq_2 \subset X_2$, then X is non-planable.*

A continuum X is said to be *triodic* (see [18], p. 262) if it contains three subcontinua A, B and C each of which does not contain any of the others, and

$$A \cap B = B \cap C = C \cap A = A \cap B \cap C$$

is a continuum. In particular, the union of three arcs pa, pb and pc each pair of which, and all three of them, having only the point p in common, is called a *simple triod*. The point p is called the *top of the triod.*

By a *disk* we mean any region D in the plane whose boundary $\bar{D} \setminus D$ is a simple closed curve.

PROPOSITION 4. *Let a continuum X contain a planable subcontinuum S and a simple triod $pa \cup pb \cup pc \subset S$. Let the arc pa be the topological limit of a sequence of arcs pa_n such that $pa_n \cap S = \{p\}$ for every $n = 1, 2, \ldots$*

Assume that for every imbedding i of S into the plane R^2 there exists a disk D in the plane such that

1° $i(p) \in D$,

2° $i(\{a, b, c\}) \subset R^2 \setminus D$, *and*

3° *if L denotes the component of $i(pb \cup pc) \cap D$ that contains the point $i(p)$ and if A denotes the component of $D \setminus L$ that intersects the arc $i(pa)$, then the point $i(p)$ is not accessible from $A \setminus i(S)$.*

Then the continuum X is non-planable.

Proof. Suppose that X is planable. Thus there exists a homeomorphism $h\colon X \to h(X) \subset R^2$. Putting $i = h|S$ we see, by the hypothesis, that there is a disk D in the plane satisfying conditions 1°, 2° and 3°. To simplify the notation we omit the imbedding i in further considerations, i.e., we write simply x in place of $i(x)$ for $x \in S$.

Let a simple closed curve C be the boundary of D. Since the point p is in the bounded component of $R^2 \setminus C$ and the point a is in the unbounded one, and since the arcs pa_n have the arc pa as their limit, we see that for almost all n the arcs pa_n intersect C. Let a_n' be (for sufficiently large n) the first point of the arc pa_n, ordered from p to a_n, which lies in C, and let a' be the similarly defined point of the arc pa. Thus, by construction, we have

$$pa_n' \setminus \{a_n'\} \subset D \quad \text{and} \quad pa' \setminus \{a'\} \subset D$$

and, by the convergence condition, almost all points a_n' must lie in the arc which is contained in C, has its end points at the end points of the arc $\bar{L}$, and which contains the point a'. Thus we have $pa_n' \cap A \neq \emptyset$. Observe that the union $C \cup \bar{L}$ is a ϑ-curve (see [14], p. 328) and that the arc

$$\bar{L} \subset pb \cup pc \subset S$$

separates D into two disks one of which is A. The arcs pa_n' have only the point p in common with S, i.e., they cannot intersect the arc $\bar{L}$ out of p. Hence, applying the theorem about the ϑ-curve (see [14], § 61, II, Theorem 2, p. 511), we conclude from $pa_n' \cap A \neq \emptyset$ and from $pa_n' \setminus \{a_n'\} \subset D$ that

$$pa_n' \setminus \{p\} \subset A.$$

But this means, by $pa_n \cap S = \{p\}$, that the point p is accessible from $A \setminus S$, contrary to 3°.

As another condition for the non-planability of a continuum, one can use the well-known triodic theorem of Moore [18] which says that each uncountable collection of triodic continua lying in the plane contains an uncountable subcollection every two elements of which have a point in common. Thus, in particular, no uncountable collection of disjoint triodic continua can exist in the plane. In other words, we have

PROPOSITION 5. *If a continuum X contains uncountably many disjoint triodic subcontinua, then X is non-planable.*

PROPOSITION 6. *Let a continuum X contain a sequence of mutually disjoint simple triods*

$$T_n = p_n a_n \cup p_n b_n \cup p_n c_n \quad (n = 0, 1, 2, \ldots),$$

where a_n, b_n, c_n are their end points and p_n is the top of T_n with

$$(1) \qquad \lim_{n\to\infty} a_n = a_0, \quad \lim_{n\to\infty} b_n = b_0, \quad \lim_{n\to\infty} c_n = c_0, \quad \lim_{n\to\infty} p_n = p_0,$$

$$(2) \qquad \operatorname{Lim}_{n\to\infty} p_n a_n = p_0 a_0, \quad \operatorname{Lim}_{n\to\infty} p_n b_n = p_0 b_0, \quad \operatorname{Lim}_{n\to\infty} p_n c_n = p_0 c_0.$$

Then X is non-planable.

Proof. Suppose that X can be imbedded in the plane R^2 under a homeomorphism $h\colon X \to h(X) \subset R^2$. Neglecting the homeomorphism h to simplify the notation, we will write x for $h(x)$, i.e., we identify X with its homeomorphic image in the plane.

Let C be a simple closed curve in R^2 such that

$$a_0 b_0 \cap C = \{a_0, b_0\}$$

and such that $a_0 b_0 \setminus \{a_0, b_0\}$ is contained in the bounded component of $R^2 \setminus C$. Thus $C \cup a_0 b_0$ is a ϑ-curve and, therefore, its complement $R^2 \setminus (C \cup a_0 b_0)$ has two bounded components D_0 and D_1 (see [14], § 61, II, Theorem 2, p. 511). Since there are infinitely many arcs $a_n b_n$, we conclude from

$$\operatorname{Lim}_{n\to\infty} a_n b_n = a_0 b_0$$

that infinitely many of them have common points with one of these components, say D_0, i.e.,

$$a_{m_n} b_{m_n} \cap D_0 \neq \emptyset$$

for some sequence of naturals m_n. Let

$$a_0' \in p_0 a_0 \setminus \{p_0, a_0\} \quad \text{and} \quad b_n' \in p_0 b_0 \setminus \{p_0, b_0\}.$$

By (2) there are points

$$a_{m_n}' \in p_{m_n} a_{m_n} \setminus \{p_{m_n}, a_{m_n}\} \quad \text{and} \quad b_{m_n}' \in p_{m_n} b_{m_n} \setminus \{p_{m_n}, b_{m_n}\}$$

such that a_{m_n}' and b_{m_n}' are in D_0 for almost all $n = 1, 2, \ldots$ Therefore, the point p_0 is not accessible from

$$D_0 \setminus \bigcup_{n=1}^{\infty} a_{m_n}' b_{m_n}',$$

whence, in particular, we conclude that $p_0c_0 \cap D_1 \neq \emptyset$. Thus there exists a point $c_0' \in p_0c_0 \setminus \{p_0, c_0\}$ such that

$$p_0c_0' \setminus \{p_0\} \subset D_1.$$

Therefore, it follows from (2) that for sufficiently large n there are points

$$c_n' \in p_nc_n \quad \text{with} \lim_{n\to\infty} c_n' = c_0'.$$

In particular, we have

$$\lim_{n\to\infty} c_{m_n}' = c_0'$$

and we see that, for sufficiently large n, the points c_{m_n}' must be in D_1. But the points p_{m_n} are in D_0 by construction. Thus, for sufficiently large n, we have

$$p_{m_n}c_{m_n}' \subset D_0 \cup a_0b_0 \cup D_1.$$

Applying the above-quoted theorem about the ϑ-curve once more we get $p_{m_n}c_{m_n}' \cap a_0b_0 \neq \emptyset$, a contradiction.

2. For curves, the imbeddability in the plane R^2 is equivalent to the imbeddability in the two-sphere. It is well known that the problem of a characterization of curves X which are non-planable is solved in the case where X is locally connected. It was firstly done for *local dendrites*, i.e., locally connected continua which contain only a finite number of simple closed curves (see [14], § 51, VII, Theorem 4, p. 303). Namely, in 1930, K. Kuratowski described two very simple graphs K_1 and K_2, called the *primitive skew graphs* (see [13], p. 272, or [14], p. 305, where these graphs are pictured), and showed that a local dendrite is non-planable if and only if it contains a homeomorphic image of either K_1 or K_2 (see [13], Theorem A, p. 278; [14], § 51, VII, Theorem 7, p. 305). Moreover, the graphs K_1 and K_2 are imbeddable in every surface except the spherical surface (see [13], Theorem C, p. 282). In general, the problem was solved in 1937 by S. Claytor who described two curves C_1 and C_2 (originally due to Kuratowski; see [8], p. 631, where these curves are pictured) and showed ([8], p. 631) that a locally connected continuum can be imbedded in the two-sphere if and only if it contains no homeomorphic image of the primitive skew graphs K_1 and K_2 or of the curves C_1 and C_2. But the problem of the planability of curves which are not locally connected is still open, and it seems to be far from solving. Only certain partial results are known (however, some of them are of great importance), e.g., each arc-like continuum is planable but the tree-like one need not be (see [1], Theorem 4 and Example 1, p. 654). In this paper we discuss some introductory properties of continua concerning this problem and we give some conditions which are sufficient (but, in general, far from being necessary) for some kind of curves to be non-planable.

3. Further parts of the paper deal mostly with *dendroids*, i.e., hereditarily unicoherent and arcwise connected continua. It is known that every dendroid is hereditarily decomposable ([3], (47), p. 239), whence it is a curve ([3], (48), p. 239) and, therefore, it can be imbedded in the Menger universal curve ([16], p. 345-360), lying in the three-dimensional cube. The methodical investigation of dendroids was initiated by Professor B. Knaster at his Topology Seminar in Wrocław about 1958. As early as in 1959 he raised the problem (see [12]) to characterize dendroids which can be imbedded in the plane R^2.

It is well known, on the one hand, that every *dendrite* (i.e., a locally connected continuum containing no simple closed curve, or — in other words — a locally connected dendroid; see [15], p. 301) is planable. This follows not only from the Kuratowski characterization of skew local dendrites but also from the fact that in the plane there exists a so-called *universal dendrite*, i.e., a dendrite which contains topologically every other dendrite (see, e.g., [16], Chapter X, 6, p. 318-322, and cf. [19], p. 57, and [11], p. 553). On the other hand, it is known that not every dendroid is planable. To see various reasons by which some dendroids cannot be imbedded in R^2 we recall first some auxiliary notions.

A point p of an arcwise connected continuum X is called a *point of order* $\mathfrak{n}$ in the classical sense if p is a common end point of exactly $\mathfrak{n}$ arcs disjoint from one another beyond p and contained in X (we write $\mathrm{Ord}_p X = \mathfrak{n}$; see [3], p. 230). In particular, a point p of X is called an *end point* of X in the classical sense if $\mathrm{Ord}_p X = 1$, and it is called a *ramification point* of X in the classical sense if $\mathrm{Ord}_p X \geqslant 3$. Henceforward, the words "in the classical sense" will be omitted. The set of all end points of X in this sense will be denoted by $E(X)$, and the set of all ramification points of X — by $R(X)$. A dendroid X is said to be a *fan* provided in X there is only one ramification point r, called the *top* of X (see [4], p. 6). If $\mathrm{Ord}_r X = \aleph_0$ or $\mathrm{Ord}_r X = \omega$, the fan X is called *countable*. In other words, the fan X is countable if and only if the set $E(X)$ is countable (see [4], p. 14).

A rather narrow class of non-planable dendroids was firstly exhibited by the first author in [3], p. 251, with the use of the notion of a ramification point. The result proved there can be formulated as

THEOREM 1. *If a dendroid X is homeomorphic to the set $R(X)$ of all ramification points of X, then X is non-planable.*

In [3], p. 245-251, there was constructed an example of a dendroid satisfying the hypothesis of Theorem 1.

A valuation of the Borel class of the sets $E(X)$ in the case where X is a planable dendroid can be used to indicate another reason of non-planability of dendroids. Namely, a Lelek's result can be reformulated as follows (see [15], § 6, Theorem, p. 307).

THEOREM 2. *If the set $E(X)$ of all end points of a dendroid X is not a $G_{\delta\sigma\delta}$-set, then X is non-planable.*

The valuation of the Borel class of sets $E(X)$ in the case where X is an arbitrary dendroid is unknown (see [15], p. 319). Moreover, we do not know any example of a dendroid X satisfying the hypothesis of Theorem 2. So the following problem seems to be open:

PROBLEM 1. Does there exist a dendroid X the set $E(X)$ of which is not a $G_{\delta\sigma\delta}$-set? (**P 1012**)

We show now some examples of dendroids the non-planability of which is a consequence of arguments used in propositions from Section 1. A large number of dendroids can be shown to be non-planable using the condition mentioned in Proposition 2. As an example one can take the well-known Borsuk's countable non-planable fan [2]. This fan contains a simple triod $T = pa \cup pb \cup pc$ and three sequences of arcs converging to the arcs ab, bc and ac, respectively. The union of T and of the arcs of the three sequences forms a planable set S with property that p is a strongly inaccessible point of S. The same Borsuk fan serves as an example of a fan whose non-planability follows from arguments mentioned in Proposition 3 or Corollary 1. As an example of a dendroid the non-planability of which is a consequence of Proposition 4 consider the following·

Take a system of cylinder coordinates (ϱ, φ, h) in the three-space R^3 and put

$$p = (0, \varphi, 0), \qquad q = (0, \varphi, 1),$$

$$a_n = \left(\frac{1}{n}, 0, 0\right), \quad b_n = \left(\frac{1}{n}, \frac{2\pi}{3}, 0\right), \quad c_n = \left(\frac{1}{n}, \frac{4\pi}{3}, 0\right),$$

$$q_n = \left(\frac{1}{n}, 0, 1\right), \quad r_n = \left(\frac{1}{n}, \frac{2\pi}{3}, 1\right), \quad s_n = \left(\frac{1}{n}, \frac{4\pi}{3}, 1\right).$$

Let xy denote the straight-line segment with end points x and y. Put

$$X = pq \cup pa_1 \cup pb_1 \cup pc_1 \cup \bigcup_{n=1}^{\infty} (a_n q_n \cup b_n r_n \cup c_n s_n).$$

It is easy to see by construction that X is a dendroid. Taking

$$S = pq \cup pb_1 \cup pc_1 \cup \bigcup_{n=1}^{\infty} (b_n r_n \cup c_n s_n)$$

we see that S is planable and $pq_n = pa_n \cup a_n q_n$, whence $pq_n \cap S = \{p\}$ for every $n = 1, 2, \ldots$ Further, for every imbedding i of S into the plane there is a plane region D satisfying 1°, 2° and 3° of Proposition 4, e.g., such one which is bounded by a simple closed curve composed of $i(r_1 b_1)$, $i(b_1 p)$, $i(pc_1)$, $i(c_1 s_1)$ and of an arc joining $i(r_1)$ with $i(s_1)$ in such a way that the arcs $i(pq)$, $i(b_n r_n)$, $i(c_n s_n)$ $(n = 2, 3, \ldots)$ without their end points lie in D. Thus $i(p)$ is inaccessible from $D \setminus i(S)$.

Examples of non-planable dendroids the non-planability of which follows from Propositions 5 and 6 will be given in Section 4.

4. A dendroid X is called *smooth* if there exists a point $p \in X$, named an *initial point* of X, such that for every convergent sequence of points a_n of X the condition

$$\lim_{n\to\infty} a_n = a$$

implies that the sequence of arcs pa_n is convergent and

$$\operatorname*{Lim}_{n\to\infty} pa_n = pa$$

(see [6], p. 298). A dendroid X is called *semi-smooth* if there exists a point $p \in X$ such that whenever a_n converges to a, then $\operatorname*{Ls}_{n\to\infty} pa_n$ is an arc (see [6], p. 306). In particular, smooth fans were investigated in [4]. It is known from [4], Theorem 9, p. 27, and [10], Corollary 4, p. 90, that every smooth fan can be imbedded into the Cantor fan, whence we have (cf. [6], p. 306)

COROLLARY 2. *Every smooth fan is planable.*

The hypothesis that the dendroid is a fan is essential in this corollary: there are examples of smooth non-planable dendroids, e.g., described in [6], p. 306 and 307.

Another example of a smooth non-planable dendroid that contains uncountably many disjoint triods (thus, the non-planability of which follows from Proposition 5) can be obtained in the following way.

Let T be the simple triod composed of three arcs pa, pb and pc with the only common point p. Let C denote the Cantor discontinuum (the Cantor ternary set). Consider a decomposition $\mathscr{D}$ of $T \times C$, the only non-degenerate element of which is the set $\{(a, \gamma)\colon \gamma \in C\}$. The decomposition space $X = T \times C/\mathscr{D}$, i.e., the image of $T \times C$ under the quotient mapping is obviously a dendroid which can be represented as the union of the Cantor set of copies of T with all end points $a \in T$ identified to one point.

Observe that one can use the harmonic sequence

$$H = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}$$

instead of the Cantor set C in the above construction to get a smooth dendroid $Y = T \times H/\mathscr{D}$ which is non-planable by Proposition 6.

Also, the hypothesis of smoothness is essential in Corollary 2: if we replace it by the semi-smoothness, then the non-planability does not follow, as it can be seen by an example of a semi-smooth non-planable fan given in [6], p. 306. The non-planability of this example follows from Proposition 2.

Let $\mathscr{A}$ be a class of spaces and let $\mathscr{P}$ be a property. Call $\mathscr{P}$ *finite* (*countable*) in the class $\mathscr{A}$ provided there is a finite (countable) set $\mathscr{F}$ of members of $\mathscr{A}$ such that a member of $\mathscr{A}$ has the property $\mathscr{P}$ if and only if it contains a homeomorphic copy of some member of $\mathscr{F}$. For example, the results of Kuratowski and of Claytor mentioned in Section 2 can be restated as follows: the property of not being imbeddable in the 2-sphere is finite in the class of local dendrites and in the class of locally connected continua. It is known that the same property is not finite in the class of dendroids (see [5]) but the following problems are still open (see [5] and [6]).

PROBLEM 2. Is the property of non-planability countable in the class of dendroids? (**P 1013**)

PROBLEM 3. Is the property of non-planability finite in the classes of 1° smooth dendroids, 2° semi-smooth dendroids, 3° semi-smooth fans? (**P 1014**)

5. It seems to be interesting to consider some relations between planability and contractibility of dendroids. Recall that a space is called *contractible* (in itself) if there exists a homotopy $H\colon X\times I\to X$, where I denotes the unit segment of reals, such that $H(\cdot, 0)$ is the identity on X and $H(\cdot, 1)$ is a constant mapping (i.e., it maps X into a point); see [14], § 54, VI, p. 374. Contractible dendroids were investigated, e.g., in [7], but no characterization of them is known. Since the smoothness of dendroids implies their contractibility (see [7], Corollary, p. 93), the examples of smooth non-planable dendroids given in Section 4 show that contractibility of dendroids does not imply their planability. However, a question arises if this implication holds for some special kinds of dendroids, e.g., for fans. In other words, a question is if there exists a contractible non-planar fan. It is our conjecture that this question has a positive answer, which can be based on the following example.

Let t be the origin of the polar coordinate system in the plane. Put in the polar coordinates (ϱ, φ),

$$p_0 = (1, 0), \quad p_n = (1, 2^{1-n}), \quad q_n = (2^{-1}, 3\cdot 2^{-(n+1)})$$

for $n = 1, 2, \ldots$ and let xy denote the straight-line segment with end, points x and y. Write

$$F = tp_0 \cup \bigcup_{n=1}^{\infty} (tp_n \cup p_n q_n).$$

The continuum F is the so-called *harmonic hooked fan* (see [4], p. 31) with the top t. It is known that F is contractible (ibidem). Let C denote, as previously, the Cantor discontinuum and let $\mathscr{D}$ be a decomposition of $F\times C$, the only non-degenerate element of which is the set $\{(t, \gamma)\colon \gamma \in C\}$. The decomposition space $X = (F\times C)/\mathscr{D}$ is obviously a fan which can be represented as the union of the Cantor set of copies of F with all tops

$t \in F$ identified to one point. Extending the homotopy which contracts F to t onto the whole X in a natural way, it is easy to verify that X is contractible. The authors do not know, however, if this particular fan is planable or not. (**P 1015**) And, which is perhaps more important, we have no general criterion to solve this problem.

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