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ON SOME REGULARITY PROPERTIES OF SOLUTIONS TO STOCHASTIC EVOLUTION EQUATIONS IN HILBERT SPACES

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0. Introduction. Let, on a separable real Hilbert space H, a linear differential equation

$$\dot{X}(t) = -AX(t)+f(t), \quad X(0) = x, \quad x \in H, \tag{1}$$

be given where A is a sectorial operator in H with the associated analytic semigroup $G(t)$, $t \geqslant 0$. It is known [17] that the Hölder-continuity of an H-valued function f implies the space-time regularity (made precise later) of a solution to equation (1).

In this paper we are concerned with similar properties of a solution to a linear stochastic differential equation

$$dX(t) = -AX(t)\,dt+dM(t), \quad X(0) = x, \quad x \in H. \tag{2}$$

We assume for the rest of the paper the following:

(a) $-A$ generates an analytic semigroup $G(t)$ on H;

(b) M is a continuous, H-valued, square-integrable martingale defined on some probability space $(\Omega, \mathscr{F}, P)$ with a given filtration $(\mathscr{F}_t)$, $t \geqslant 0$.

Assumption (b) will be relaxed only at the end of Section 3 where M is allowed to be a cylindrical process.

By a *solution to equation* (2) we mean the so-called mild solution defined by the formula

$$X(t) = G(t)x+\int_0^t G(t-s)\,dM(s). \tag{3}$$

It is well known that if a strong solution to (2) exists, then it is of the form (3) (see [1]).

In order to introduce the notion of regularity let us recall the definition of the scale of spaces H_α, $\alpha \geqslant 0$ (see [6]): H_α is the domain of the operator $A_1^{\alpha/2}$,

$$H_\alpha = D(A_1^{\alpha/2}),$$

endowed with the norm

$$\|x\|_\alpha = \|A_1^{\alpha/2} x\|, \quad x \in H_\alpha,$$

where $A_1 = A + I\beta$, and β is a positive real number which belongs to the resolvent set of the generator $-A$. For the definitions and properties of fractional powers of operators see, e.g., [15]. It will be shown that to some extent the same regularity results hold for solution (3) to equation (2) as for the deterministic equation (1). Namely, it will be shown (with some additional assumptions) that the mild solution (3) is Hölder-continuous as a process taking values in the spaces H_α. Results of this type can be interesting for themselves. They also give a better understanding of the more interesting nonlinear stochastic equations of the semilinear type:

$$dX(t) = [-AX(t) + f(X(t))]\,dt + dM(t),$$

where a nonlinear function f is not defined on the whole space H. Such equations are considered in [6]–[8]. Since the martingale M can be a white noise process in some spaces H_α, the regularity results for the process (3) are of some importance for developing stochastic differential equations with white noise perturbation. Equation (2) does not make sense in H_α but it can be solved in H and the solution is a "good" stochastic process in H_α (see [11]).

The problem of regularity of a mild solution to (2) arouse some interest recently. The continuity of (3) in H for the semigroups of contractions was proved by Kotelenez [9] and Ichikawa [7]. A more general result of this type can be found in the recent work of Da Prato et al. [3]. For analytic semigroups, the regularity of a mild solution as a process in H_α was investigated by Kotelenez [11], Ichikawa [7] and also in [3]. Similar problems were considered by Da Prato (e.g., in [2]). Linear stochastic equations with white noise perturbations were treated in [4] and [12].

1. Auxiliary results. The following lemma is a generalization of a result presented in [1]:

LEMMA 1. *Let H, G, K be separable Hilbert spaces, $F \in \Lambda^2(H, G, M)$. Let U be a closed operator defined on the domain $D(U) \subset G$,*

$$U\colon D(U) \to K \quad \text{and} \quad UF \in \Lambda^2(H, K, M)$$

(and thus the composition $UFQ_M^{1/2}$ makes sense). Then

$$\int_0^t F\,dM \in D(U) \quad \text{and} \quad U\int_0^t F\,dM = \int_0^t UF\,dM.$$

Moreover, $F \in \Lambda^2(H, D, M)$, where $D = D(U)$ is endowed with the graph norm.

Remark. For all the notation and facts concerning stochastic integrals in a Hilbert space see [13].

For the proof of this lemma we need a selection-type result proved by Zabczyk (unpublished).

LEMMA 2. *Let $(\Omega, \mathscr{F})$ be a given measurable space and let $\mathscr{K}$ denote the space of all nonnegative (and thus selfadjoint), compact operators on a separable Hilbert space H. Assume that $\mathscr{A}\colon \Omega \to \mathscr{K}$ is a measurable transformation in the sense that, for arbitrary $x, y \in H$, $\omega \to \langle \mathscr{A}(\omega)x, y\rangle$ is a real-valued measurable transformation. Then there exist a sequence of functions $\lambda_1(\omega) \geqslant \lambda_2(\omega) \geqslant \ldots \geqslant 0$ and a sequence $x_1(\omega), x_2(\omega), \ldots$ of vectors such that*

1° $\|x_i(\omega)\| \leqslant 1$, $i = 1, 2, \ldots$, $\omega \in \Omega$;

2° *if* $\lambda_i(\omega) \neq 0$, *then* $\|x_i(\omega)\| = 1$;

3° *if* $i \neq j$ *and* $\lambda_i(\omega) > 0$, $\lambda_j(\omega) > 0$, *then*

$$\langle x_i(\omega), x_j(\omega)\rangle = 0;$$

4° *the functions* λ_i *and* x_i *are measurable for arbitrary* i;

5° *for arbitrary* $x \in H$ *and* $\omega \in \Omega$,

$$\mathscr{A}(\omega)x = \sum_{i=1}^{\infty} \lambda_i(\omega)\langle x, x_i(\omega)\rangle x_i(\omega)$$

and

$$\|\mathscr{A}(\omega)x\|^2 = \sum_{i=1}^{\infty} \lambda_i^2(\omega)|\langle x, x_i(\omega)\rangle|^2.$$

Proof of Lemma 1. Let us notice that the lemma is obvious if U is a bounded operator. To prove the general case we use Lemma 2. The process Q_M associated with the martingale M can be decomposed into the form

$$Q_M x = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i\rangle e_i, \quad x \in H,$$

where λ_i, e_i are predictable processes with properties 1°–5° of Lemma 2 and the sum is taken, for every ω, over those i's for which $\lambda_i(s, \omega) > 0$. Now let us define the process $R_n(s, \omega)$ as the orthogonal projection on the linear span of those $e_i(s, \omega)$ for which $\lambda_i(s, \omega) > 0$, $i \leqslant n$. Then, for every $x \in H$, $R_n x$ is a predictable process. Let $F_n = FR_n$; then $F_n(s, \omega)$ is a bounded operator and $F_n x$ is a predictable process for every $x \in H$. We have also

$$\|F_n\|_{\Lambda^2} \leqslant \|F\|_{\Lambda^2} < \infty,$$

which yields $F_n \in \Lambda^2(H, G, M)$. Obviously, we have

$$\|(F_n - F)Q^{1/2}\|_{HS}^2 = \sum_{k=1}^{\infty} \lambda_k \|(F_n - F)e_k\|^2 = \sum_{k=n+1}^{\infty} \lambda_k \|Fe_k\|^2.$$

In addition, since $Q_M^{1/2}$ and R_n commute, we get

$$\|(F_n - F)Q^{1/2}\|_{HS} = \|FQ^{1/2}(R_n - I)\|_{HS} \leqslant \|FQ^{1/2}\|_{HS}.$$

Thus, by the dominated convergence theorem, we obtain

$$\|F_n - F\|_{\Lambda^2} \to 0.$$

By the same reasoning, $UF_n \in \Lambda^2(H, K, M)$, and in this space $UF_n \to UF$. These facts imply that $F_n \in \Lambda^2(H, D, M)$ and

$$F_n \to F \quad \text{in } \Lambda^2(H, D, M).$$

Since U is bounded on D, we get

$$U \int F_n dM = \int UF_n dM.$$

Now by the above results and the closedness of U the assertion follows.

The next lemma is a simple consequence of a more general result given in [14], but we give here a direct proof based on the Itô lemma.

LEMMA 3. *Let* $F \in \Lambda^2(H, K, M)$ *and*

$$N(t) = \int_0^t F(s)\, dM(s).$$

Then for every $p \geqslant 1$ *there exists* $C_p > 0$ *such that*

$$\mathrm{E}\,\|N(t)\|^{2p} \leqslant C_p \mathrm{E}\langle N\rangle_t^p$$

and C_p *is a deterministic constant.*

Proof. Let ϕ: $K \to R$ be of the form

$$\phi(x) = \|x\|^{2p}.$$

Then we have

$$\|\phi''(x)\| \leqslant A_p \|x\|^{2p-2}.$$

Then the Itô lemma (see [13]) yields

$$\|N(t)\|^{2p} = \int_0^t \phi'(N(S))\, dM(s) + \tfrac{1}{2}\int_0^t \phi''(N(s))\, d\langle\langle M\rangle\rangle_s,$$

and thereby

$$\begin{aligned}\mathrm{E}\,\|N(t)\|^{2p} &\leqslant \tfrac{1}{2}\mathrm{E}\int_0^t \mathrm{tr}[\phi''(N(s))\, Q_N(s)]\, d\langle N\rangle_s \\ &\leqslant \tfrac{1}{2}\mathrm{E}[\langle N\rangle_t \sup_{s\leqslant t} \|\phi''(N(s))\|] \leqslant \tfrac{1}{2}A_p \mathrm{E}(\sup_{s\leqslant t}\|N(s)\|^{2p-2}\langle N\rangle_t) \\ &\leqslant \tfrac{1}{2}A_p(\mathrm{E}\,\|N(t)\|^{2p})^{1-1/p}(\mathrm{E}\langle N\rangle_t^p)^{1/p}.\end{aligned}$$

Now it is easy to see that

$$\mathrm{E}\,\|N(t)\|^{2p} \leqslant (\tfrac{1}{2}A_p)^p\, \mathrm{E}\langle N\rangle_t^p.$$

2. Moments of a mild solution. In this part of the paper some results on the existence of the moments of the mild solution (3) are presented. We begin with

THEOREM 1. *Let X be a mild solution to equation* (2), $X(0) = 0$. *If for some* $\alpha \geq 0$ *and* $t \geq 0$

$$(4) \qquad \sup_{t' \leq t} \int_0^{t'} \frac{d\langle M\rangle_s}{(t'-s)^\alpha} < \infty \ \text{a.s.},$$

then $X(t) \in H_\alpha$ *a.s. Moreover, if for some* $p \geq 1$

$$(5) \qquad \mathrm{E}\left(\int_0^t \frac{d\langle M\rangle_s}{(t-s)^\alpha}\right)^p < \infty,$$

then $\mathrm{E}\|X(t)\|_\alpha^{2p} < \infty$.

For the proof of this theorem and the next ones we need some facts from the theory of analytic semigroups. We collect them in

LEMMA 4 ([15]). *Let* $-A$ *generate an analytic semigroup* $G(t)$, $t \geq 0$, *on a Banach space* B *and let zero belong to the resolvent set of* A. *Then*

(1) *for every* $t > 0$ *and* $\alpha \geq 0$,

$$G(t)B \subset D(A^\alpha);$$

(2) *for every* $x \in D(A^\alpha)$, $\alpha \geq 0$,

$$G(t)A^\alpha x = A^\alpha G(t)x;$$

(3) *for every* $t > 0$ *and* $\alpha \geq 0$, *the operator* $A^\alpha G(t)$ *is bounded and*

$$\|A^\alpha G(t)\| \leq L_\alpha t^{-\alpha};$$

(4) *for* $0 \leq \alpha \leq 1$ *and* $x \in D(A^\alpha)$,

$$\|G(t)x - x\| \leq D_\alpha t^\alpha \|A^\alpha x\|, \qquad t \in \langle 0, T\rangle;$$

(5) *for* $\alpha \geq 0$, $\beta \geq 0$ *and* $x \in D(A^{\alpha+\beta})$,

$$A^\beta x \in D(A^\alpha).$$

In all the proofs that follow it is assumed that zero belongs to the resolvent set of A; the general case can be deduced easily by shifting the generator $-A$ and simultaneous change of the martingale.

P r o o f o f T h e o r e m 1. We begin with the second part of the theorem, so assume that (5) is fulfilled. By means of Lemmas 1, 3 and 4 we get

$$\begin{aligned}
\mathrm{E}\|X(t)\|_\alpha^{2p} &= \mathrm{E}\left\|\int_0^t A^{\alpha/2}G(t-s)\,dM(s)\right\|^{2p} \\
&\leq C_p \mathrm{E}\Big(\int_0^t \|A^{\alpha/2}G(t-s)Q_M^{1/2}(s)\|_{HS}^2\, d\langle M\rangle_s\Big)^p \\
&\leq C_p L_\alpha^{2p} \mathrm{E}\left(\int_0^t \frac{d\langle M\rangle_s}{(t-s)^\alpha}\right)^p < \infty,
\end{aligned}$$

which proves the second part of the theorem. If

$$\mathrm{E}\int_0^t \frac{d\langle M\rangle_s}{(t-s)^\alpha} < \infty,$$

then $X(t) \in H_\alpha$. If only (4) is fulfilled, then define the sequence of stopping times (T_n):

$$T_n = \begin{cases} \inf\left\{t \leqslant T;\ \sup\limits_{t' \leqslant t} \int_0^{t'} \frac{d\langle M\rangle_s}{(t'-s)^\alpha} > n\right\} & \text{if such a } t \text{ exists,} \\ T & \text{if } \int_0^t \frac{d\langle M\rangle_s}{(t-s)^\alpha} \leqslant n \text{ for all } t \leqslant T. \end{cases}$$

T_n's are Markov times, $T_n \leqslant T_{n+1}$ and, for almost every ω, $T_n(\omega) = T$ for n sufficiently large. Let

$$M^n(t) = M(t \wedge T_n) \quad \text{and} \quad X^n(t) = \int_0^t G(t-s)\, dM^n(s).$$

This definition implies that $\mathrm{E}\|X^n(t)\|_\alpha^2 < \infty$, and thus $X^n(t) \in H_\alpha$. Since $X(t) = X^n(t)$ for $t < T_n$, we obtain the desired result.

Now we consider the case where A is selfadjoint. In this case Lemma 2 and the spectral theorem ([16], p. 227) yield

$$\mathrm{E}\|X(t)\|_\alpha^2 = \mathrm{E}\int_0^\infty \lambda^\alpha \Big(\int_0^t e^{-2\lambda(t-s)} d\langle M\rangle_s\Big)\mu(d\lambda), \tag{6}$$

where

$$\mu(d\lambda) = \sum_{k=1}^\infty \lambda_k \langle P(d\lambda)e_k, e_k\rangle, \tag{7}$$

$P(d\lambda)$ is the spectral measure of the operator A, and $\lambda_k = \lambda_k(s, \omega)$, $e_k = e_k(s, \omega)$ arise from the predictable decomposition of the operator Q_M given by Lemma 1. Formula (6) allows us to construct some counterexamples and to obtain more exact results in the selfadjoint case.

EXAMPLE. It can be shown that for every $\alpha > 0$ one can find a martingale M and a selfadjoint A such that $\mathrm{E}\|X(t)\|_\alpha^2 = \infty$. It is enough to consider the martingale

$$M(t) = e_1 \int_0^t f(s)\, dW(s),$$

where e_1 is a fixed vector, $\|e_1\| = 1$, W is the Wiener process on H, and

$$f(s) = \begin{cases} (t_0-s)^{-\beta} & \text{for } s < t_0, \\ 1 & \text{for } s \geqslant t_0, \end{cases}$$

where $0 < t_0 \leqslant T$, $2\beta < 1$. Then

$$\mu(d\lambda) = \langle P(d\lambda)e_1, e_1\rangle$$

and

$$\mathrm{E}\,\|X(t_0)\|_\alpha^2 = \int_0^\infty \lambda^{\alpha+2\beta-1}\Big(\int_0^{\lambda t_0} s^{-2\beta}e^{-2s}\,ds\Big)\mu(d\lambda) = \infty,$$

provided that $\alpha+2\beta > 1$ and the operator A is properly chosen. Since $X(t)$ is a Gaussian process, we see also that $X(t_0)$ is not a random variable in H_α.

It can be shown that for some cases condition (5) is also necessary for the finiteness of the second moment of the process X.

PROPOSITION. *Assume that the operator Q_M is independent of (s, ω). If for every selfadjoint semigroup $G(t)$*

$$\mathrm{E}\,\|X(t)\|_\alpha^2 < \infty,$$

then

$$\mathrm{E}\int_0^t \frac{d\langle M\rangle_s}{(t-s)^\alpha} < \infty.$$

For the proof we will need the following

LEMMA 5. *Let (e_i) be a CONS in a Hilbert space H and let the sequence (λ_i) such that*

$$\lambda_i \geqslant 0, \qquad \sum_{i=1}^{\infty} \lambda_i = 1$$

be given. Then for every probability measure on R absolutely continuous with respect to the Lebesgue measure there exists a selfadjoint operator A such that

$$\mu(d\lambda) = \sum_{k=1}^{\infty} \lambda_k \langle P(d\lambda)e_k, e_k\rangle,$$

where $P(d\lambda)$ is the spectral decomposition of the operator A.

The proof of this lemma follows easily from the spectral theorem (see [16]).

Proof of the Proposition. By assumption we have

$$\int_0^\infty \lambda^\alpha\, \mathrm{E}\int_0^t e^{-2\lambda(t-s)}\,d\langle M\rangle_s\,\mu(d\lambda) < \infty$$

for every selfadjoint positive A, and thus, by the above lemma, for every probability measure μ on $\langle 0, \infty)$ absolutely continuous with respect to the Lebesgue measure. Thereby the functional T defined by

$$T(g) = \int_0^\infty \Big[\mathrm{E}\int_0^t \lambda^\alpha e^{-2\lambda(t-s)}\,d\langle M\rangle_s\Big]\,g(\lambda)\,d\lambda$$

is finite for every $g \geqslant 0$ such that

$$\int_0^\infty g(\lambda)\,d\lambda = 1.$$

This implies that T is finite, and thus continuous on $L^1(0, \infty)$. So we get

$$\sup_{\lambda \geqslant 0} \mathrm{E} \int_0^t \lambda^\alpha e^{-2\lambda(t-s)} d\langle M\rangle_s < \infty.$$

It can be easily shown that the function

$$G(\lambda) = \mathrm{E} \int_0^t \lambda^\alpha e^{-2\lambda(t-s)} d\langle M\rangle_s$$

is differentiable for $\lambda > 0$ and one can differentiate the integrand. Hence

$$\sup_{\lambda \geqslant 0} G(\lambda) = \left(\frac{\alpha}{2e}\right)^\alpha \mathrm{E} \int_0^t \frac{d\langle M\rangle_s}{(t-s)^\alpha} < \infty,$$

which completes the proof.

Remark. If M is a Wiener process, then the assumptions of Theorem 1 are fulfilled for every α, $0 \leqslant \alpha < 1$, and every $p \geqslant 1$. Formula (6) yields in addition that $\mathrm{E}\|X(t)\|_1^2 < \infty$, and thus $X(t) \in D(A^{1/2})$ for $0 \leqslant t \leqslant T$. Properties of the solution X in $D(A^{1/2})$ for A selfadjoint were recently investigated more thoroughly in [3]. For $\alpha > 1$ the generator $-A$ can be chosen such that $\mathrm{E}\|X(t)\|_\alpha^2 = \infty$.

3. Regularity of a mild solution. We will prove here that the mild solution to equation (2) is Hölder-continuous in the spaces H_α.

THEOREM 2. *Assume that the martingale M fulfills the condition*

$$d\langle M\rangle_s \leqslant Q(s)\,ds,$$

where Q is a real progressively measurable process such that for some $p > 1$

$$\int_0^T Q^p(s)\,ds < \infty \quad a.s. \tag{8}$$

Then for every $\alpha \in \langle 0, 1-1/p)$ the solution $X(0) = 0$ to equation (2) *is Hölder-continuous in H_α with any exponent a such that*

$$a < (1-\alpha)/2 - 1/(2p).$$

Remark. Assumption (8) implies that the process $\langle M\rangle$ is Hölder-continuous, and thus, by [10], so is the martingale M. For such martingales a better result was obtained in [3]. If the martingale M is a Wiener process, then we obtain the same result as in [3].

Proof. We assume, as in the proof of Theorem 1, that zero belongs to the resolvent set of the operator A. The proof is by the Kolmogorov theorem [5]. Let us take $0 \leqslant t_1 < t_2 \leqslant T$. Assume first that, for every $n \geqslant 1$,

$$\text{(9)} \qquad \mathrm{E}\Big(\int_0^T Q^p(s)\,ds\Big)^n < \infty .$$

Then, by Lemma 1,

$$\begin{aligned}
&\mathrm{E}\,\|X(t_2)-X(t_1)\|_\alpha^{2n} \\
&\quad = \mathrm{E}\,\Big\|\int_0^{t_2} A^{\alpha/2}G(t_2-s)\,dM(s)-\int_0^{t_1} A^{\alpha/2}G(t_1-s)\,dM(s)\Big\|^{2n} \\
&\quad \leqslant 2^{2n-1}\mathrm{E}\,\Big\|\int_0^{t_1} A^{\alpha/2}\big(G(t_2-s)-G(t_1-s)\big)\,dM(s)\Big\|^{2n} \\
&\qquad + 2^{2n-1}\mathrm{E}\,\Big\|\int_{t_1}^{t_2} A^{\alpha/2}G(t_2-s)\,dM(s)\Big\|^{2n}.
\end{aligned}$$

Let us consider the first term. By Lemmas 3 and 4 we get

$$\begin{aligned}
&\mathrm{E}\,\Big\|\int_0^{t_1} A^{\alpha/2}\big(G(t_2-s)-G(t_1-s)\big)\,dM(s)\Big\|^{2n} \\
&\quad \leqslant C_n\mathrm{E}\Big(\int_0^{t_1} \|A^{\alpha/2}\big(G(t_2-s)-G(t_1-s)\big)Q_M^{1/2}\|_{HS}^2\,d\langle M\rangle_s\Big)^n \\
&\quad \leqslant C_n\mathrm{E}\Big(\int_0^{t_1}\sum_{k=1}^{\infty} \|A^{\alpha/2}\big(G(t_2-t_1)-I\big)G(t_1-s)Q_M^{1/2}e_k\|^2\,d\langle M\rangle_s\Big)^n \\
&\quad \leqslant C_nD(t_2-t_1)^{n(\alpha+2\beta)}\mathrm{E}\Big(\int_0^{t_1}\sum_{k=1}^{\infty} \|A^{\beta+\alpha/2}G(t_1-s)Q_M^{1/2}e_k\|^2\,d\langle M\rangle_s\Big)^n \\
&\quad \leqslant C_nDL(t_2-t_1)^{n(\alpha+2\beta)}\mathrm{E}\left(\int_0^{t_1}\frac{d\langle M\rangle_s}{(t_1-s)^{\alpha+2\beta}}\right)^n,
\end{aligned}$$

where $D = D_{\alpha/2+\beta}^{2n}$, $L = L_{\alpha/2+\beta}^{2n}$, and (e_k) is an orthonormal base in H. By the Hölder inequality,

$$\mathrm{E}\left(\int_0^{t_1}\frac{d\langle M\rangle_s}{(t_1-s)^{\alpha+2\beta}}\right)^n \leqslant \mathrm{E}\Big(\int_0^{t_1} Q^p(s)\,ds\Big)^{n/p}\Big(\int_0^{t_1}(t_1-s)^{-q(\alpha+2\beta)}\,ds\Big)^{n/q}$$

for $1/q = 1-1/p$, and the last expression is finite if β is such that

$$\alpha+2\beta < 1-1/p.$$

The second term can be estimated similarly:

$$\mathrm{E}\,\Big\|\int_{t_1}^{t_2} A^{\alpha/2}G(t_2-s)\,dM(s)\Big\|^{2n} \leqslant C_n\mathrm{E}\Big(\int_{t_1}^{t_2}\|A^{\alpha/2}G(t_2-s)\|^2\,d\langle M\rangle_s\Big)^n$$

$$\leqslant C_n L' \mathrm{E}\Big(\int_{t_1}^{t_2} \frac{Q(s)}{(t_2-s)^{\alpha}}\Big)^n \leqslant C_n L' (\int_{t_1}^{t_2} Q^p(s)\,ds)^{n/p} (\int_{t_1}^{t_2} (t_2-s)^{-\alpha q}\,ds)^{n/q}$$

$$\leqslant C_n L' \mathrm{E}(\int_0^T Q^p(s)\,ds)^{n/p} \Big(\frac{1}{1-\alpha q}\Big)^{n/q} (t_2-t_1)^{(1-\alpha q)n/q},$$

where $L' = L_{\alpha/2}^{2n}$. These inequalities taken together yield

$$\mathrm{E}\,\|X(t_2)-X(t_1)\|^{2n} \leqslant B_1(t_2-t_1)^{n(\alpha+2\beta)} + B_2(t_2-t_1)^{n(1-\alpha-1/p)}.$$

From the Kolmogorov theorem it follows that X is Hölder-continuous in H_α with an arbitrary exponent smaller than

$$(1-\alpha)/2 - 1/(2p) - 1/(2n)$$

and, in fact, since n can be taken arbitrarily large, a can be smaller than

$$(1-\alpha)/2 - 1/(2p).$$

Thus the theorem is proved when (9) is satisfied. The general case can be reduced to the previous one. Let us define a sequence of stopping times

$$T_n = \begin{cases} \inf\{t \leqslant T;\ \int_0^t Q^p(s)\,ds \geqslant n\} & \text{if such a } t \text{ exists,} \\ T & \text{if } \int_0^T Q(s)\,ds < n. \end{cases}$$

Since the process Q is progressively measurable, we have

$$\{T_n \leqslant t\} \in \mathscr{F}_t$$

and, by (8), for almost every ω, $T_n(\omega) = T$ for n sufficiently large.

Now, let

$$M^n(t) = M(t \wedge T_n) \quad \text{and} \quad X^n(t) = \int_0^t G(t-s)\,dM^n(s).$$

Then, by the first part of the proof, the processes X^n are Hölder-continuous for all n with an arbitrary exponent smaller than $(1-\alpha)/2-1/(2p)$. Since

$$XI_{\langle 0,T_n\rangle} = X^n I_{\langle 0,T_n\rangle},$$

we get the required property.

A similar theorem can be proved in a more general situation. Namely, assume now that M is a weakly continuous 2-cylindrical martingale with finite quadratic variation (see [13]). Under the above assumptions there exists a finite quadratic Doléans measure μ_M with bounded variation $|\mu_M|$. Assume also that the semigroup $G(t)$ has the following radonifying property:

$(\exists \gamma \in \langle 0, 1))$ $A^{-\gamma/2}$ is a Hilbert–Schmidt operator.

Then it can be shown (see [13]) that the meaning can be given to the stochastic integral

$$(10) \qquad X(t) = \int_0^t G(t-s)\,dM(s)$$

and it is a well-defined stochastic process. For this process the following generalization of Theorem 2 can be proved:

THEOREM 2′. *Assume that the variation measure $|\mu_M|$ of the quadratic Doléans measure μ_M is absolutely continuous with respect to Lebesgue measure and*

$$d|\mu_M| \leqslant Q(s)\,ds \ \text{a.s.},$$

where Q is a real progressively measurable process such that

$$\int_0^T Q^p(s)\,ds < \infty \ \text{a.s.} \quad \text{for some } p > 1.$$

Then for every $\alpha \in \langle 0, 1-1/p-\gamma)$ the process X defined by (10) *is Hölder-continuous in H_α with arbitrary exponent a such that*

$$a < \frac{1-(\alpha+\gamma)}{2} - \frac{1}{2p}.$$

The proof is similar to the proof of Theorem 2.

Remark. If the cylindrical martingale M is a cylindrical Wiener process, then the assumptions of Theorem 2′ are obviously fulfilled for every $p \geqslant 1$ and we obtain the same result as in [4]. In the general case our assumptions are weaker than those of [12] and the results obtained are stronger.

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REFERENCES

[1] A. Chojnowska-Michalik, *Stochastic differential equations in Hilbert spaces*, pp. 53–74 in: *Probability Theory* (Warsaw 1976), Banach Center Publ. 5, Polish Scientific Publishers, Warsaw 1979.

[2] G. Da Prato, *Maximal regularity for stochastic convolutions and applications to stochastic evolution equations in Hilbert spaces*, pp. 41–52 in: L. Arnold and P. Kotelenez (Eds.), *Stochastic Space-Time Models and Limit Theorems*, D. Reidel Publishing Company, Dordrecht 1985.

[3] — S. Kwapień and J. Zabczyk, *Regularity of solutions of linear stochastic equations in Hilbert spaces*, Stochastics 23 (1987), pp. 1–23.

[4] G. Da Prato and J. Zabczyk, *A note on semilinear stochastic equations*, Differential and Integral Equations 1 (1988), pp. 143–155.

[5] I. I. Gihman and A. V. Skorohod, *The Theory of Stochastic Processes, I*, Springer-Verlag, New York–Heidelberg 1974.
[6] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer-Verlag, 1981.
[7] A. Ichikawa, *Semilinear stochastic evolution equations. Boundedness, stability and invariant measures*, Stochastics 12 (1984), pp. 1–39.
[8] — *A semigroup model for parabolic equations with boundary and pointwise noise*, pp. 81–94 in: L. Arnold and P. Kotelenez (Eds.), *Stochastic Space-Time Models and Limit Theorems*, D. Reidel Publishing Company, Dordrecht 1985.
[9] P. Kotelenez, *A submartingale type inequality with applications to stochastic evolution equations*, Stochastics 8 (1982), pp. 139–151.
[10] — *Continuity properties of Hilbert space valued martingales*, Stochastic Process. Appl. 17 (1984), pp. 115–125.
[11] — *On the semigroup approach to stochastic evolution equations*, pp. 95–140 in: L. Arnold and P. Kotelenez (Eds.), *Stochastic Space-Time Models and Limit Theorems*, D. Reidel Publishing Company, Dordrecht 1985.
[12] — *A maximal inequality for stochastic convolution integrals on Hilbert spaces and space-time regularity of linear stochastic partial differential equations*, Stochastics 21 (1987), pp. 345–358.
[13] M. Metivier and J. Pellanmail, *Stochastic Integration*, Academic Press, New York 1980.
[14] E. Pardoux, *Intégrales stochastiques hilbertiennes*, Cah. Math. Déc. no 7617, Université Paris IX, 1976.
[15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York–Berlin 1983.
[16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 1, Academic Press, New York 1980.
[17] E. Sinestrari, *On the abstract Cauchy problem of parabolic type in spaces of continuous functions;* J. Math. Anal. Appl. 107 (1985), pp. 16–66.

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