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NON-SIDON SETS IN THE SUPPORT OF A FOURIER-STIELTJES TRANSFORM

BY

COLIN C. GRAHAM (EVANSTON, ILLINOIS)

1. Introduction. Several years ago, Hartman and Ryll-Nardzewski ([4], Problem 711) asked whether there exists a continuous measure μ on the circle group T such that, for some $\varepsilon > 0$, $\{n\colon |\hat{\mu}(n)| > \varepsilon\} = E(\mu, \varepsilon)$ is not a Sidon set. An affirmative answer was first provided by Kaufman [7]. Kaufman used an elegant category argument to produce a class of examples. Later, the present author showed in [3] that if μ were concentrated on a Kronecker set (see [9] for a definition), then $E(\mu, \varepsilon)$ contained arbitrarily long arithmetic progressions for all $0 < \varepsilon < \|\mu\|$. A different class of examples was provided by Ramirez [8] who showed that Riesz products μ had the property that $E(\mu, \varepsilon)$ was not a Sidon set for $0 < \varepsilon \leqslant \limsup |\hat{\mu}(n)|^2$. Izuchi studies in [5] the class $\mathscr{S}$ of measures μ on an LCA group G such that $E(\mu, \varepsilon)$ is a Sidon set for all $\varepsilon > 0$. He shows, in particular, that $\mathscr{S}$ is an L-ideal [12] of measures, and that $\mathscr{S}$ contains (for compact G, of course) the measures $M_0(G)$ such that $\hat{\mu}$ vanishes at infinity on the dual Γ of G.

This note improves the method of [3] to obtain the observation of [8]: $E(\mu, \varepsilon)$ contains arbitrarily "large squares" (see below) if $\hat{\mu}$ does not vanish at infinity on Γ. An (obvious modification of the) argument of Salinger and Varopoulos ([10], proof of Theorem 3) shows that if $E(\mu, \varepsilon)$ contains arbitrarily large squares, then $E(\mu, \varepsilon)$ is a set of analyticity, and so is not a Sidon set. (That $E(\mu, \varepsilon)$ is not Sidon also follows from [6], p. 54.) Thus, the class $\mathscr{S}$ of measures studied in [5] coincides with $M_0(G)$ for compact G. The result of this note may also be considered as a partial converse to Drury's theorem [2]. For another related result, see Blei [1].

We now define our terms and state our theorems.

Definition. An *n-square* in the LCA group Γ is a set of the form AB, where A, B are subsets of Γ of cardinality n. A set $X \subseteq \Gamma$ *contains arbitrarily large squares* if X contains an n-square for $n = 1, 2, \ldots$ (The reader will have noted that Γ is written as a multiplicative group. This (unusual) convention will simplify the notation later on.)

The reader should be warned that an n-square may not have cardinality n^2. However, Salinger and Varopoulos [10] show that if X contains arbitrarily large squares, then X contains n-squares which do have n^2 elements for $n = 1, 2, \ldots$

For a regular Borel measure μ on the LCA group G (with dual Γ) and $\varepsilon > 0$, we define $E(\mu, \varepsilon)$ by

$$E(\mu, \varepsilon) = \{\gamma \in \Gamma \colon |\hat{\mu}(\gamma)| > \varepsilon\}, \tag{1}$$

where $\hat{\mu}$ denotes the Fourier-Stieltjes fransform of μ.

In this note we prove

THEOREM 1. *Let G be a non-discrete LCA group with dual Γ. Let μ be a regular Borel measure on G such that the Fourier-Stieltjes transform $\hat{\mu}$ of μ does not vanish at infinity on Γ. Then there exists $\varepsilon > 0$ such that*

$$E(\mu, \varepsilon) = \{\gamma \in \Gamma \colon |\hat{\mu}(\gamma)| \geqslant \varepsilon\}$$

contains arbitrarily large squares.

COROLLARY 1. *If G, Γ, μ and ε satisfy the hypotheses and conclusion of Theorem 1, then $E(\mu, \varepsilon)$ is a set of analyticity.*

COROLLARY 2. *Let G be an infinite LCA group with dual Γ. Let μ be a regular Borel measure on G. If $E(\mu, \varepsilon)$ is a Sidon set for all $\varepsilon > 0$, then $\hat{\mu}$ vanishes at infinity on Γ.*

2. Proofs. Corollary 2 follows at once from Corollary 1, and Corollary 1 follows from (the proof of) Theorem 3 of [10].

We now actually prove a stronger result than Theorem 1:

THEOREM 1′. *Let G and μ satisfy the hypotheses of Theorem 1. Let $m \geqslant 2$ and $n \geqslant 1$. Then there exists $\varepsilon > 0$ such that for each $m \geqslant 1$ there exist sets $A_1, \ldots, A_m \subseteq \Gamma$, each having cardinality n, such that*

$$A_1 A_2 \ldots A_m \subseteq E(\mu, \varepsilon).$$

The method of [10] shows that $A_1, A_2, \ldots, A_m$ may be chosen so that $A_1 A_2 \ldots A_m$ has cardinality n^m.

For the proof of the theorem, we use the generalized character notions of Šreider [11]. The reader might consult Taylor [12] for many results about generalized characters. Finally, let us point out again that we write Γ as a multiplicative group in the proof below.

Fix $m \geqslant 1$. We have several steps.

(A) We may assume that Γ is discrete, since $\hat{\mu}(\gamma) \neq 0$ and Γ not discrete implies that, for some neighborhoods (infinite sets in particular) U, V of 0 contained in Γ,

$$E(\mu, \varepsilon) \supseteq U \cdot V \cdot \gamma \quad \text{for some } \varepsilon > 0.$$

(B) We may assume that G is metrizable. Indeed, let $\{\gamma_j\}_1^\infty$ be an infinite subset of Γ such that $\limsup|\hat{\mu}(\gamma_j)| \neq 0$. Let Λ be the subgroup of Γ generated by $\{\gamma_j\}_1^\infty$.

It will be sufficient to find our sets $A_1, \ldots, A_m \subseteq \Lambda$. A moment's thought will convince the reader that we may therefore assume $\Gamma = \Lambda$.

(C) Let $\{\gamma_j\}_1^\infty$ be as in (B). We may assume (G being metrizable) that $\lim \gamma_j = f$ exists in $L^\infty(\mu)$, the limit being weak-*, of course. Then $f \not\equiv 0$ a.e. $d\mu$, since

$$\int f d\mu = \lim \int \gamma_j d\mu \neq 0.$$

Therefore, f^m (pointwise product in $L^\infty(\mu)$) is not identically zero a.e. $d\mu$, so the measure defined by $g \to \int g f^m d\mu$ ($g \in C(G)$) is not identically zero. Therefore, there exists $\gamma_0 \in \Gamma$ such that

$$\int f^m \gamma_0 d\mu \neq 0.$$

We define ε by

$$\varepsilon = \frac{1}{2}\left|\int f^m \gamma_0 d\mu\right|. \tag{2}$$

(D) Fix $n \geqslant 1$. Since f is the weak-* limit of $\{\gamma_j\}$, there exist distinct $\varrho_{1,1}, \ldots, \varrho_{1,n} \in \{\gamma_j\}_1^\infty$ such that

$$\left|\int \varrho_{1,j} f^{m-1} \gamma_0 d\mu\right| > \left(\frac{1}{2}\right)^{1/m} \left|\int f^m \gamma_0 d\mu\right| \quad (1 \leqslant j \leqslant n). \tag{3}$$

(E) We now induct. Suppose that distinct $\varrho_{l,1}, \ldots, \varrho_{l,n} \in \{\gamma_j\}_1^\infty$ have been found for $l = 1, 2, \ldots, k-1 < m$ such that

$$\left|\int \varrho_{1,j(1)} \cdots \varrho_{l,j(l)} f^{m-l} \gamma_0 d\mu\right| > \left(\frac{1}{2}\right)^{l/m} \left|\int f^m \gamma_0 d\mu\right|, \tag{4}$$

where $1 \leqslant j(1), \ldots, j(l) \leqslant n$, and $1 \leqslant l \leqslant k-1$.

We now find $\varrho_{k,1}, \ldots, \varrho_{k,n}$. Since $\lim \gamma_j = f$ weak-* in $L^\infty(\mu)$, there exist distinct $\varrho_{k,1}, \ldots, \varrho_{k,n} \in \{\gamma_j\}_1^\infty$ such that

$$\begin{aligned} &\left|\int \varrho_{1,j(1)} \cdots \varrho_{k,j(k)} f^{m-k} \gamma_0 d\mu\right| \\ &\qquad > \left(\frac{1}{2}\right)^{1/m} \left|\int \varrho_{1,j(1)} \cdots \varrho_{k-1,j(k-1)} f^{m-k+1} \gamma_0 d\mu\right|. \end{aligned} \tag{5}$$

Now (4) (with $l = k-1$) and (5) yield (4) with $l = k$. This completes the induction, i.e., (4) now holds for $1 \leqslant j(1), \ldots, j(m) \leqslant n$ (and $m = l$).

By the choice (formula (2)) of ε, we see that (4) (for $l = m$) implies

$$(6) \qquad \left|\int \varrho_{1,j(1)} \cdots \varrho_{m,j(m)} \gamma_0 d\mu\right| > \varepsilon \quad (1 \leqslant j(1), \ldots, j(m) \leqslant n).$$

Setting

$$A_l = \{\varrho_{l,1}, \ldots, \varrho_{l,n}\} \ (1 \leqslant l < m) \quad \text{and} \quad A_m = \{\varrho_{m,1}\gamma_0, \ldots, \varrho_{m,n}\gamma_0\},$$

we see that $A_1 A_2 \ldots A_m \subseteq E(\mu, \varepsilon)$ as required. This completes the proof of Theorem 1'.

3. Remarks.

(i) It is easy to see, using Riesz products, for example, that there is no necessary relationship between m, ε and $\limsup |\hat{\mu}(\gamma)|$.

(ii) That the A_i can be chosen so that $A_1 A_2 \ldots A_m$ has cardinality n^m cs simply done in our situation. Indeed, assume that $A_1 A_2 \ldots A_{k-1}$ has iardinality n^{k-1} $(1 < k \leqslant m)$. Since $\lim \gamma_j = f$ weak-*, there is a J such that $j \geqslant J$ implies (5) whenever $\varrho_{k,j(k)} = \gamma_j$ (and $A_i = \{\varrho_{i,1}, \ldots, \varrho_{i,n}\}$ for $1 < i < k$). We choose $\varrho_{k,1} = \gamma_j$, $j \geqslant J$ arbitrarily. Then there must exist, in the infinite set $\{\gamma_i \colon i > j\}$, a $\varrho_{k,2} = \gamma_i$ such that

$$[A_1 A_2 \ldots A_{k-i}\{\varrho_{k,1}\}] \cap A_1 A_2 \ldots A_{k-i}\{\varrho_{k,2}\} = \emptyset.$$

A simple induction now shows that $A_1 A_2 \ldots A_m$ may be chosen with cardinality n^m.

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