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ON DECOMPOSABILITY SEMIGROUPS ON THE REAL LINE

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Preliminaries. Let P be a probability measure on the real line R and let $\hat{P}$ be the characteristic function of P, i.e. the Fourier transform of P:

$$\hat{P}(t) = \int_{-\infty}^{\infty} e^{itx} P(dx).$$

For every $s \in R$ we shall denote by T_s the mapping $T_s x = sx$ ($x \in R$). Further, $T_s P$ will denote the measure defined by the formula $T_s P(E) = P(T_s^{-1}(E))$ for all Borel subsets E of R. The decomposability semigroup $D(P)$ corresponding to P consists of all real numbers s for which there exists a probability measure P_s such that $P = T_s P * P_s$. The semigroup operation is simply the multiplication of numbers. The concept of the decomposability semigroup associated with probability measures has been introduced by Urbanik in [4]. It has been also proved there that some probability properties of measures correspond to algebraic and topological properties of their decomposability semigroups.

It is evident that $s \in D(P)$ if and only if there exists a probability measure P_s such that $\hat{P}(t) = \hat{P}(st)\hat{P}_s(t)$ ($t \in R$). It is well known that P is non-degenerate if and only if $D(P)$ is compact (see [4], Theorem 1). In other words, for a non-degenerate P, $D(P)$ is a compact subsemigroup of the multiplicative semigroup $[-1, 1]$. Further, it is clear that always $0 \in D(P)$ and $1 \in D(P)$. Urbanik raised the problem whether this condition characterizes decomposability semigroups among compact ones. It is easy to prove that $-1 \in D(P)$ if and only if P is a translation of a symmetric probability measure. The theory of self-decomposable probability measures affords important examples of decomposability semigroups. Namely, Lévy's characterization of non-degenerate self-decomposable laws P is equivalent to the inclusion $[0, 1] \subset D(P)$ (see [2], Section 23.3). Hence, in particular, we get the following statement: a non-degenerate probability measure P is a translation of a symmetric self-decomposable one if and only if $D(P) = [-1, 1]$.

Some non-trivial examples of decomposability semigroups for probability measures have been given in [3] and [5]. Let S be a symmetric subsemigroup of $[-1,1]$ containing both elements 0 and 1 and satisfying the condition

$$\sum_{s\in S} s^2 < \infty.$$

Urbanik proved in [5] that for every number q from the interval $[-1,0]$ there exists a probability measure P such that $S\cap[q,1] = D(P)$. Even on the real line the problem of characterization of those semigroups which are decomposability semigroups $D(P)$ for probability measures is still open. We show in Section 2 that the set of decomposability semigroups $D(P)$ corresponding to the non-degenerate measures P is "dense" in the set of all compact subsemigroups of $[-1,1]$ containing both elements 0 and 1.

Now we define some subsemigroups of the decomposability semigroup $D(P)$. The decomposability semigroup $D^{\mathrm{id}}(P)$ $(D^{\mathrm{sd}}(P))$ corresponding to the infinitely divisible (self-decomposable) measure P consists of all numbers s for which there exists an infinitely divisible (self-decomposable) measure P_s such that $P = T_sP*P_s$. The essential topological and algebraic properties of the semigroups $D^{\mathrm{id}}(P)$ and $D^{\mathrm{sd}}(P)$ are the same as those of $D(P)$.

Urbanik proved in [5] that for a stable measure P we have the equality $D(P) = D^{\mathrm{id}}(P) = D^{\mathrm{sd}}(P)$. It is easy to give an example of the self-decomposable measure P for which $D^{\mathrm{id}}(P) \neq D^{\mathrm{sd}}(P)$. We prove in Section 1 that there exists an infinitely divisible measure P for which $D(P) \neq D^{\mathrm{id}}(P)$. In Sections 1 and 3 we give a characterization of decomposability semigroups $D^{\mathrm{id}}(P)$ and $D^{\mathrm{sd}}(P)$.

1. The decomposability semigroup $D^{\mathrm{id}}(P)$. Adopting the notation

$$\int_{-x}^{x}{}^{*} \varphi(u)\,dN(u) \qquad (0 < x \leqslant \infty)$$

for the sum of two Stieltjes integrals

$$\int_{-x}^{-0} \varphi(u)\,dN(u) + \int_{+0}^{x} \varphi(u)\,dN(u)$$

we can write down the well-known necessary and sufficient condition for a function f to be the characteristic function of an infinitely divisible measure P in the following form (see, e.g., [2]):

$$\log f(t) = Mit - \frac{Gt^2}{2} + \int_{-\infty}^{\infty}{}^{*} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) dN(u),$$

where $M \in R$, $G \in R^+$, $N(u) = N_1(u)$ for $u < 0$ and $N(u) = N_2(u)$ for $u > 0$, N_1 and N_2 being non-decreasing functions such that

$$\int_{-a}^{a} {}^{*}\, u^2 dN(u) < \infty, \qquad 0 < a < \infty.$$

The function $N(u)$ $(u \in (-\infty, 0)\cup(0, \infty))$ is called a *spectral function*. The function N generates the measure ν on $(-\infty, 0)\cup(0, \infty)$ by the formula $\nu\big((a, b]\big) = N(b+)-N(a+)$ $(a < b,\ 0 \notin (a, b])$. The measure ν is called a *spectral measure*.

It is evident that $s \in D^{\mathrm{id}}(P)$ if and only if there exists an infinitely divisible measure P_s such that $\hat{P}(t)/\hat{P}(st)$ $(t \in R)$ is the characteristic function of P_s. Thus, if $0 < s < 1$, then $s \in D^{\mathrm{id}}(P)$ if and only if the functions $N_1^s(u) = N_1(u)-N_1(u/s)$ for $u < 0$ and $N_2^s(u) = N_2(u)-N_2(u/s)$ for $u > 0$ are non-decreasing. If $-1 < s < 0$, then $s \in D^{\mathrm{id}}(P)$ if and only if the functions $N_1^s(u) = N_1(u)+N_2(u/s)$ for $u < 0$ and $N_2^s(u) = N_2(u)+ +N_1(u/s)$ for $u > 0$ are non-decreasing. It is clear that if $N_1 \equiv \mathrm{const}$ and $N_2 \not\equiv \mathrm{const}$, then $(-1, 0)\cap D^{\mathrm{id}}(P) = \emptyset$.

THEOREM 1.1. *There exists an infinitely divisible measure such that* $D^{\mathrm{id}}(P) \neq D(P)$.

Proof. Let $G(u) = 2u$ $(u \in [1/2, 1])$, $B > 0$. There exists a number a from a certain interval $I \subset [1/2, 1]$ (see [1], p. 157) such that $G_1(a)- -G_1(a-h) > ch$, where $0 < h < h^{(3)}$, $G_1(w) = G(u_0 w)$, $h^{(3)} > 0$, $u_0 \in [b, 1]$ $(0 < b < 1)$, $c > 0$. If we put

$$E = \left\{\frac{k}{2^n} : n = 2, 3, \ldots;\ k = 2^{n-1}+1, \ldots, 2^n-1;\ (k, 2^n) = 1\right\},$$

then $I \cap E$ is dense in I. Since $G_1(a)-G_1(a-h) = 2u_0 h$, we may choose the number a from E, i.e. there exist integers k and n such that $(k, 2^n) = 1$ and $a = k/2^n$.

Now, there exists a number $s > 0$ such that

$$g_1(t) = \exp\Big(-Bt^2+\int_{1/2}^{1} (e^{itu}-1)\,dG(u)-s(e^{it/2^n}-1)\Big)$$

is a characteristic function (see [1], p. 130 and 161). Since $\exp\big(s(e^{it/2}-1)\big)$ is also a characteristic function, the function $g(t)$ given by the formula

$$g(t) = g_1(t)\exp\big(s(e^{it/2}-1)\big)$$

is the characteristic function of a certain measure Q. We write $g(t)$ in the form

$$g(t) = \exp\left(Ait-Bt^2+\int_{+0}^{\infty}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)dN_2(u)\right),$$

where

$$N_2(u) = -\chi_{(0,1/2^n]} - (1+s)\chi_{(1/2^n,1/2]} + (2u-2)\chi_{(1/2,1]} \qquad (u > 0).$$

Since N_2 is not non-decreasing, Q is not infinitely divisible.

Let f be a function defined by the formula

$$f(t) = \prod_{k=0}^{\infty} g\left(\frac{1}{2^k} t\right).$$

The function f is the characteristic function of a certain infinitely divisible measure P for which the corresponding spectral function M is given by the formula

$$\begin{aligned} M_2(u) &= \sum_{k=0}^{\infty} N_2(2^k u) \\ &= \sum_{k=0}^{n-2} (2^{k+1}u + ks)\chi_{[1/2^{k+1},1/2^k)} + \sum_{k=n-1}^{\infty} \big(2^{k+1}u + (n-1)s\big)\chi_{[1/2^{k+1},1/2^k)} \\ &\qquad\qquad (u > 0), \end{aligned}$$

$$M_1 \equiv 0 \qquad (u < 0).$$

It is evident that $1/2 \in D(P)$ $(P_{1/2} = Q)$. Since Q is not infinitely divisible, $1/2 \notin D^{\mathrm{id}}(P)$. This completes the proof.

Let B denote the set of all infinitely divisible measures P for which the corresponding spectral measure ν has an atom (in other words, the corresponding spectral function N is non-continuous). Now we give a characterization of the decomposability semigroups $D^{\mathrm{id}}(P)$ for which the corresponding infinitely divisible measure P belongs to B.

THEOREM 1.2. *Let $S \subset [-1,1]$ be a compact semigroup containing 0 and 1. Then there exists a measure $P \in B$ such that $D^{\mathrm{id}}(P) = S$ if and only if*

$$\sum_{s\in S} s^2 < \infty. \tag{1.1}$$

Proof. Necessity. Assume that a measure P with the spectral measure ν belongs to B and $D^{\mathrm{id}}(P) = S$. We shall use the following obvious inequalities:

$$\nu(\{x\}) = T_s\nu(\{sx\}) \qquad (x \neq 0,\ s \in [-1,1]), \tag{1.2}$$

$$T_s\nu(\{x\}) \leqslant \nu(\{x\}) \qquad (x \neq 0,\ s \in D^{\mathrm{id}}(P)). \tag{1.3}$$

If $s \in D^{\mathrm{id}}(P)$ $(s \neq 0)$ and $\nu(\{x\}) > 0$, then by (1.2) and (1.3) we have

$$\nu(\{sx\}) > 0 \quad \text{and} \quad \nu(\{x\}) \leqslant \nu(\{sx\}),$$

and hence

$$\infty > \int_{-x}^{x} {}^{*} \mathring{u}^2 \nu(du) \geqslant \sum_{s\in S} (sx)^2 \nu(\{sx\}) \geqslant x^2 \nu(\{x\}) \sum_{s\in S} s^2.$$

Thus (1.1) holds.

Sufficiency. Let us suppose that (1.1) holds. For each point x ($x \in R$) we shall denote by δ_x the probability measure concentrated at the point x. Let P be an infinitely divisible measure for which the corresponding spectral measure ν is equal to $\sum_{s\in S\setminus\{0\}} \delta_s$. We show that $D^{\mathrm{id}}(P) = S$.

It is evident that if $-1 \in S$, then $-1 \in D^{\mathrm{id}}(P)$. If $s_0 \in S$ and $0 < |s_0| < 1$, then $s_0 S \subset S$ and P_{s_0} is an infinitely divisible measure with the spectral measure

$$\nu_{s_0} = \sum_{s\in(S\setminus s_0 S)} \delta_s.$$

Thus $S \subset D^{\mathrm{id}}(P)$.

Conversely, if $s \in D^{\mathrm{id}}(P)$, then, by (1.2) and (1.3), $\nu(\{s\}) \geqslant \nu(\{1\})$. Since $\nu(\{1\}) > 0$, we have $s \in S$. Thus $D^{\mathrm{id}}(P) \subset S$. This completes the proof.

Let $N(u)$ $(u \in (-\infty, 0) \cup (0, \infty))$ be the spectral function corresponding to an infinitely divisible measure. We introduce the functions $\bar{N}_i(\bar{u}) = N_i(u)$ $(i = 1, 2)$, where $\bar{u} = \log u$ for $u > 0$ and $\bar{u} = -\log(-u)$ for $u < 0$. Since

$$\frac{d\bar{N}_i(\bar{u})}{d\bar{u}} = |u| \frac{dN_i(u)}{du},$$

$N_i(u)$ is non-decreasing if and only if $\bar{N}_i(\bar{u})$ is non-decreasing $(i = 1, 2)$. If $s > 0$, then

$$\overline{N_1^s}(\bar{u}) = \bar{N}_1(u) - \bar{N}_1(\bar{u} + \bar{s}), \qquad \overline{N_2^s}(\bar{u}) = \bar{N}_2(\bar{u}) - \bar{N}_2(\bar{u} - \bar{s})$$

and

$$\overline{N_1^{-s}}(\bar{u}) = \bar{N}_1(\bar{u}) + \bar{N}_2(-\bar{u} - \bar{s}), \qquad \overline{N_2^{-s}}(\bar{u}) = \bar{N}_2(\bar{u}) + \bar{N}_1(-\bar{u} + \bar{s}).$$

THEOREM 1.3. *Let S be a compact semigroup containing* 0 *and* 1. *Then there exists an infinitely divisible measure P such that $S = D^{\mathrm{id}}(P)$.*

Proof. Clearly, there exist an $n \in \{1, 2, \ldots, \infty\}$ and disjoint open intervals $U_k = (u_k^L, u_k^R)$ $(0 \leqslant k < n+1, u_0^R = 1)$ such that

$$[-1, 1] \setminus S = \bigcup_{k=0}^{n} U_k \cup (\{-1\} \setminus S).$$

If $0 \leqslant k < n+1$, then there exists a compact semigroup S_k such that $S \subset S_k$ and

$$[-1, 1] \setminus S_k = \bigcup_{j=0}^{n_k} U_{k,j} \cup \bigcup_{j=0}^{m_k} W_{k,j} \cup (\{-1\} \setminus S),$$

where $m_k < \infty$, $n_k < \infty$, and

$$U_{k,j} = (u_{k,j}^L, u_{k,j}^R) \subset [-1, 0] \quad (j = 0, 1, \ldots, n_k)$$

and

$$W_{k,j} = (w_{k,j}^L, w_{k,j}^R) \subset [0, 1] \quad (j = 0, 1, \ldots, m_k)$$

are disjoint open intervals such that

$$u_{k,j}^R < u_{k,j+1}^R \ (j = 0, 1, \ldots, n_k - 1), \qquad w_{k,j+1}^R < w_{k,j}^R \ (j = 0, 1, \ldots, m_k - 1),$$

$$W_{k,0} = U_0, \quad W_{k,m_k} = U_k \quad \text{or} \quad U_{k,n_k} = U_k.$$

By induction we define a sequence $\{x_k\}_{k=0}^{n}$. Let $x_0 = 1$, and if we have $\{x_j\}_{j=0}^{k-1}$ $(k < n)$, then we put

$$x_k = x_{k-1} \min\{-u_{k-1,n_{k-1}}^R, w_{k-1,m_{k-1}}^L\}.$$

If $n < \infty$, then we put $x_{n+1} = 0$.

Further, let P be an infinitely divisible measure for which the corresponding spectral function is continuous ($N_2(1) = N_1(-1) = 0$) and

$$\frac{d\overline{N}_2(\overline{u})}{d\overline{u}} = \sum_{k=1}^{n} (a_{k-1}\chi_{A_k} + a_k \chi_{B_k}), \qquad \frac{d\overline{N}_1(\overline{u})}{d\overline{u}} = \sum_{k=1}^{n} (a_{k-1}\chi_{C_k} + a_k \chi_{D_k}),$$

where $0 \leqslant a_k < a_{k+1} \leqslant 1$ $(0 \leqslant k < n+1)$, and

$$A_k = \overline{x}_k + \Big(\bigcup_{j=0}^{m_k} (\overline{w}_{k,j}^L, \overline{w}_{k,j}^R - b_k)\Big), \qquad B_k = (\overline{x}_{k+1}, \overline{x}_k] \setminus A_k,$$

$$C_k = -\overline{x}_k + \Big(\bigcup_{j=1}^{n_k} (\overline{u}_{k,j}^L + b_k, \overline{u}_{k,j}^R) \cup F_k\Big), \qquad F_k = \begin{cases} (b_k, \overline{u}_{k,0}^R) & \text{if } -1 \in S_k, \\ \overline{U}_{k,0} & \text{if } -1 \notin S_k, \end{cases}$$

$$D_k = [-\overline{x}_k, -\overline{x}_{k+1}) \setminus C_k,$$

$$b_k = \frac{1}{2} \min\{ \min_{j=0,\ldots,n_k} |\overline{u}_{k,j}^R - \overline{u}_{k,j}^L|, \min_{j=0,\ldots,m_k} |\overline{w}_{k,j}^R - \overline{w}_{k,j}^L|\}.$$

It is evident that such a measure exists.

Now we shall show that for every k $(0 \leqslant k < n+1)$ the functions $\overline{N_1^s}(\overline{u}) | [-\overline{x}_k, -\overline{x}_{k+1})$ and $\overline{N_2^s}(\overline{u}) | (\overline{x}_{k+1}, \overline{x}_k]$ are non-decreasing if and only if $s \in S_k$. Since

$$\bigcap_{k=0}^{n} S_k = S,$$

we have $D^{\mathrm{id}}(P) = S$.

Let $0 \leqslant k < n+1$. We first prove that if $\overline{N_1^s}(\overline{u}) | [-\overline{x}_k, \overline{x}_{k+1})$ and $\overline{N_2^s}(\overline{u}) | (\overline{x}_{k+1}, \overline{x}_k]$ are non-decreasing, then $s \in S_k$. We use the obvious inequalities

$$(1.4)\qquad \frac{d\overline{N_2^s}(\bar{u})}{d\bar{u}} = (a_{k-1}-a_k)\chi_1 < 0 \quad \text{for } \bar{u} \in \overline{W}_{k,i}+\bar{x}_k,\ s \in W_{k,i},\quad i = 0, 1, \ldots, m_k,$$

$$(1.5)\qquad \frac{d\overline{N_1^s}(\bar{u})}{d\bar{u}} = (a_{k-1}-a_k)\chi_2 < 0 \quad \text{for } \bar{u} \in \overline{U}_{k,i}-\bar{x}_k,\ s \in U_{k,i},\quad i = 0, 1, \ldots, n_k,$$

where χ_1 is the indicator function of the set

$$[\bar{x}_k - b_k + \bar{s},\ \bar{x}_k + \bar{s}] \cap (\overline{W}_{k,i} + \bar{x}_k),$$

χ_2 is the indicator function of the set

$$[-\bar{x}_k - (\overline{-s}),\ -\bar{x}_k + b_k - (\overline{-s})] \cap (\overline{U}_{k,i} - \bar{x}_k),$$

and $\overline{U}_{k,i} = (\bar{u}_{k,i}^L, \bar{u}_{k,i}^R)$, $\overline{W}_{k,i} = (\bar{w}_{k,i}^L, \bar{w}_{k,i}^R)$.

Suppose that $s \notin S_k$. If

$$s \in \bigcup_{i=0}^{m_k} W_{k,i} = [0, 1[\setminus S_k,$$

then by (1.4) the function $\overline{N_2^s}(\bar{u}) | (\bar{x}_{k+1}, \bar{x}_k]$ is not non-decreasing ($\overline{W}_{k,i} + \bar{x}_k \subset A_k$, $i = 0, 1, \ldots, m_k$). If

$$s \in \bigcup_{i=0}^{n_k} U_{k,i} = (-1, 0) \setminus S_k,$$

then by (1.5) the function $\overline{N_1^s}(\bar{u}) | [-\bar{x}_k, -\bar{x}_{k+1})$ is not non-decreasing ($\overline{U}_{k,i} - \bar{x}_k \subset C_k$, $i = 0, 1, \ldots, n_k$). If $-1 \notin S_k$, then $\overline{N_1^{-1}}(\bar{u}) | [-\bar{x}_k, -\bar{x}_{k+1})$ is not non-decreasing since

$$\frac{d\overline{N_1^{-1}}(\bar{u})}{d\bar{u}}\bigg|_{(-\bar{x}_k, -\bar{x}_k+b_k)} = a_{k-1} - a_k < 0 \quad ((-\bar{x}_k, -\bar{x}_k + b_k) \subset C_k).$$

Thus, if $\overline{N_1^s}(\bar{u}) | [-\bar{x}_k, -\bar{x}_{k+1})$ and $\overline{N_2^s}(\bar{u}) | (\bar{x}_{k+1}, \bar{x}_k]$ are non-decreasing, then $s \in S_k$.

Now, we shall show that if $s \in S_k$, then $\overline{N_1^s}(\bar{u}) | [-\bar{x}_k, -\bar{x}_{k+1})$ and $\overline{N_2^s}(\bar{u}) | (\bar{x}_{k+1}, \bar{x}_k]$ are non-decreasing. We write S_k as $S_k = (-S_k^-) \cup S_k^+$, where if $s \in S_k^- \cup S_k^+$, then $s \geqslant 0$. If $s \in S_k^+$, then we have the inclusions

$$(1.6)\qquad \left\{\bar{u} : \frac{d\overline{N}_2(\bar{u}-\bar{s})}{d\bar{u}} \geqslant a_k\right\} = \big(B_k \cup (-\infty, \bar{x}_{k+1})\big) + \bar{s} \subset (-\infty, \bar{x}_{k+1}) \cup B_k$$

and

$$(1.7)\qquad \left\{\bar{u} : \frac{d\overline{N}_1(\bar{u}+\bar{s})}{d\bar{u}} \geqslant a_k\right\} = \big(D_k \cup (-\bar{x}_{k+1}, \infty)\big) - \bar{s} \subset C_k \cup (-\bar{x}_{k+1}, \infty),$$

since $S_k^+ \cdot S_k^+ \subset S_k^+$ and $S_k^- \cdot S_k^+ \subset S_k^-$, respectively. Similarly, for $s \in S_k^-$ we have the inclusions

$$(1.8)\qquad \left\{\bar{u} : \frac{d\bar{N}_1(-\bar{u}+\bar{s})}{d\bar{u}} \leqslant -a_k\right\}$$
$$= \big(-D_k \cup (-\infty, \bar{x}_{k+1})\big) + \bar{s} \subset B_k \cup (-\infty, \bar{x}_{k+1}),$$

$$(1.9)\qquad \left\{\bar{u} : \frac{d\bar{N}_2(-\bar{u}-\bar{s})}{d\bar{u}} \leqslant -a_k\right\}$$
$$= \big(-B_k \cup (-\bar{x}_{k+1}, \infty)\big) - \bar{s} \subset D_k \cup (-\bar{x}_{k+1}, \infty).$$

We can write N_i^s in the form $N_i^s = N_i - (N_i - N_i^s)$ $(i = 1, 2)$. Let $s \in S_k$. If $\bar{u} \in (-1)^i(\bar{x}_{k+1}, \bar{x}_k]$, then

$$\frac{d\big(\bar{N}_i(\bar{u}) - \overline{N_i^s}(\bar{u})\big)}{d\bar{u}} \leqslant a_{k-1} \quad \text{or} \quad \frac{d\big(\bar{N}_i(\bar{u}) - \overline{N_i^s}(\bar{u})\big)}{d\bar{u}} = a_k.$$

Since

$$\left.\frac{d\bar{N}_i(\bar{u})}{d\bar{u}}\right|_{(-1)^i(\bar{x}_{k+1}, \bar{x}_k]} \geqslant a_{k-1},$$

in the first case we have the inequality

$$\frac{d\overline{N_i^s}(\bar{u})}{d\bar{u}} \geqslant 0.$$

If $\bar{u} \in (-1)^i(\bar{x}_{k+1}, \bar{x}_k]$, then by (1.6)-(1.9) the equality

$$\frac{d\big(\bar{N}_i(\bar{u}) - \overline{N_i^s}(\bar{u})\big)}{d\bar{u}} = a_k$$

implies

$$\frac{d\bar{N}_i(\bar{u})}{d\bar{u}} = a_k;$$

thus in the second case we have

$$\frac{d\overline{N_i^s}(\bar{u})}{d\bar{u}} \geqslant 0.$$

Hence, if $s \in S_k$, then $\overline{N_i^s}(\bar{u}) \,|\, (-1)^i(\bar{x}_{k+1}, \bar{x}_k]$ is non-decreasing $(i = 1, 2)$. This completes the proof.

It is clear from the proof of Theorem 1.3 that for every compact semigroup S containing 0 and 1 there exists a probability measure P for which the corresponding spectral function is continuous and such that $S = D^{\mathrm{id}}(P)$.

2. The decomposability semigroup $D(P)$. It follows from the forthcoming Lemma 2.1 and from Theorem 1.3 that the decomposability semigroups $D(P)$, where P is a non-degenerate probability measure, are "dense" in the set of all compact subsemigroups of the multiplicative semigroup $[-1, 1]$ containing 0 and 1.

LEMMA 2.1. *Let P be an infinitely divisible measure; then*

$$\bigcap_{n=1}^{\infty} D(P_n) = D^{\mathrm{id}}(P), \qquad \text{where } P = P_n^{*n},$$

and there exists a subsequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $D(P_{n_k}) \supset D(P_{n_{k+1}})$ and

$$\bigcap_{k=1}^{\infty} D(P_{n_k}) = D^{\mathrm{id}}(P).$$

Proof. First we prove that

$$D^{\mathrm{id}}(P) \subseteq \bigcap_{n=1}^{\infty} D(P_n).$$

If $N(u)$ is a spectral function corresponding to P, then $N(u)/n$ is a spectral function corresponding to P_n, thus $D^{\mathrm{id}}(P) = D^{\mathrm{id}}(P_n)$. Since $D^{\mathrm{id}}(P_n) \subseteq D(P_n)$ for any integer n,

$$D^{\mathrm{id}}(P) \subseteq \bigcap_{n=1}^{\infty} D(P_n).$$

Let now

$$s \in \bigcap_{n=1}^{\infty} D(P_n),$$

i.e. for any integer n we have $P_n = T_s P_n * P_{s_n}$, for a certain probability measure P_{s_n}. Thus for every n we obtain the equality

$$P = (T_s P_n)^{*n} * P_{s_n}^{*n} = T_s P * P_s,$$

where $P_s = P_{s_n}^{*n}$ and $s \in D^{\mathrm{id}}(P)$. This completes the proof of the first part.

For a sequence $\{n_k\}_{k=1}^{\infty}$ it suffices to take $\{2^k\}_{k=1}^{\infty}$.

THEOREM 2.1. *Let S be a compact semigroup containing 0 and 1 and let K be a compact set such that K and S are disjoint. Then there exists a probability measure P such that*

$$S \subset D(P) \subset [-1, 1] \setminus K.$$

Proof. By Theorem 1.3 there exists an infinitely divisible measure Q such that $D^{\mathrm{id}}(Q) = S$. By Lemma 2.1 there exists a sequence of infinitely divisible measures $\{P_k\}_{k=1}^{\infty}$ such that

$$D(P_{k+1}) \subset D(P_k) \quad \text{and} \quad \bigcap_{k=1}^{\infty} D(P_k) = D^{\mathrm{id}}(Q).$$

For any $x \in K$ there exists an integer $n(x)$ such that $x \notin D(P_k)$ for every $k \geqslant n(x)$. Since $(-1, 1]\setminus D(P_k)$ is an open set, there exists a neighborhood $U(x)$ of x such that $U(x) \cap D(P_k) = \emptyset$ for all $k \geqslant n(x)$. Since

$$K \subset \bigcup_{x \in K} U(x)$$

and K is compact, there exists a finite subset $\{x_j\}_{j=1}^m \subset K$ ($m < \infty$) such that

$$K \subset \bigcup_{j=1,\ldots,m} U(x_j).$$

Thus

$$K \cap D(P_k) = \emptyset \quad \text{for all } k \geqslant n = \max_{j=1,\ldots,m} n(x_j).$$

For a measure P it suffices to take $P = P_n$. This completes the proof.

3. The decomposability semigroup $D^{sd}(P)$. Let E denote the set of all self-decomposable measures P for which the derivative $N'(u)$ of the corresponding spectral function $N(u)$ is non-continuous ($N'(u)$ denotes a left-hand or right-hand derivative). Now we give a characterization of the decomposability semigroups $D^{sd}(P)$ for which the corresponding self-decomposable measure P belongs to E.

THEOREM 3.1. *Let S be a compact semigroup containing* 0 *and* 1. *Then there exists a measure $P \in E$ such that $D^{sd}(P) = S$ if and only if* (1.1) *holds.*

Proof. It is well known that the infinitely divisible measure P for which N is a spectral function is self-decomposable if and only if $N(u)$ has left-hand and right-hand derivatives for $u \neq 0$ and $-uN'(u)$ is non-decreasing on $(-\infty, 0)$ and on $(0, \infty)$, where $N'(u)$ denotes a left-hand or right-hand derivative ([2], p. 324).

Necessity. Let P belong to E, let N be a spectral function corresponding to P and $D^{sd}(P) = S$. Without loss of generality we may assume that $N_2'(u)$ $(u > 0)$ is non-continuous. Let M denote the function defined by $M(u) = -uN'(u)$, $M_1(u) = -uN_1'(u)$ $(u < 0)$ and $M_2(u) = -uN_2'(u)$ $(u > 0)$. Since $P \in E$, $M_1(u)$ $(u < 0)$ and $M_2(u)$ $(u > 0)$ are non-decreasing and $M_1 \geqslant 0$, $M_2 \leqslant 0$. If $0 < |s| < 1$, then $s \in D^{sd}(P)$ if and only if $M_i^s(u)$ $(i = 1, 2)$ are non-decreasing. Let $x > 0$ be a discontinuity point of $N_2'(u)$. We put $h = M_2(x+) - M_2(x-)$. Since $h > 0$, so if $s \in D^{sd}(P)$ $(s \neq 0)$, then $M(u)$ is non-continuous in sx and $M(sx+) - M(sx-) \geqslant h$. Since $M_1(u)$ and $M_2(u)$ are non-decreasing, $M(u)$ is non-continuous on a countable set without cluster points in $R\setminus\{0\}$. Thus we can write S in the form

$$S = \{s_k'\}_{k=1}^m \cup \{s_k\}_{k=1}^n \cup \{0\}, \tag{3.1}$$

where $\{s_k'\}_{k=1}^m \overset{\text{df}}{=} \emptyset$ if $S \cap [-1, 0) = \emptyset$, $s_k' < s_{k+1}' < 0$ $(1 \leqslant k < m)$ and $0 < s_k < s_{k+1}$ $(1 \leqslant k < m)$.

If $S \cap (0,1) \neq \emptyset$, then

$$(3.2)\quad \int_0^x u^2 dN_2(u) = \int_0^x u u N_2'(u) du \geqslant \int_0^x u \sum_{k=1}^{\infty} kh\chi_{(s_{k+1}x, s_k x]} du = \frac{hx^2}{2} \sum_{k=1}^{\infty} s_k^2.$$

If $S \cap (-1, 0) \neq \emptyset$, then

$$(3.3)\quad \int_{-x}^{0} u^2 dN_1(u) = \int_{-x}^{0} \Big(-u\big(-uN_1'(u)\big)\Big) du$$

$$\geqslant \int_{-x}^{0} \Big(-u \sum_{k=1}^{\infty} kh\chi_{[s_k' x, s_{k+1}' x)}\Big) du = \frac{hx^2}{2} \sum_{k=1}^{\infty} s_k'^2.$$

Since

$$h > 0 \quad \text{and} \quad \int_{-x}^{x} u^2 dN(u) < \infty,$$

by (3.2) and (3.3) we obtain (1.1). This completes the proof of the necessity.

Sufficiency. Let us suppose that (1.1) holds. We write S in the form (3.1). Let P be an infinitely divisible measure for which the corresponding spectral function N is continuous ($N(-2) = N(2) = 0$) and

$$-uN'(u) = \sum_{k=1}^{m} k\chi_{[s_k', s_{k+1}')} - \sum_{k=1}^{n} k\chi_{(s_{k+1}, s_k]}$$

(if $m = 1$, then we put $s_2' = 0$, and if $n = 1$, then we put $s_2 = 0$). Similarly as in Theorem 1.2 we can show that $D^{\mathrm{sd}}(P) = S$. This completes the proof.

THEOREM 3.2. *Let S be a compact semigroup containing* 0 *and* 1. *Then there exists a self-decomposable measure Q such that $S = D^{\mathrm{sd}}(Q)$.*

Proof. Let Q be an infinitely divisible measure for which M is the corresponding spectral function. Since

$$u\frac{dM_i(u)}{du} = (-1)^i \frac{d\bar{M}_i(\bar{u})}{d\bar{u}},$$

Q is self-decomposable if and only if the functions

$$(-1)^{i+1} \frac{d\bar{M}_i(\bar{u})}{d\bar{u}} \quad (i = 1, 2)$$

are non-decreasing.

Let P be an infinitely divisible measure for which $D^{\mathrm{id}}(P) = S$ and the corresponding spectral function N is given as in the proof of Theorem 1.3. Let M be a continuous function on $R \setminus \{0\}$ ($M(-\infty) = M(\infty) = 0$) such that

$$(-1)^{i+1} \frac{d\bar{M}_i(\bar{u})}{d\bar{u}} = \bar{N}_i(\bar{u}) \quad (i = 1, 2),$$

where $M_1 = M|(-\infty, 0)$ and $M_2 = M|(0, \infty)$. We have the inequality

$$\int_{-1}^{1}{}^{*} u^2 dM(u) = -\int_{-1}^{1}{}^{*} u\bar{N}(\bar{u})du \leqslant -2\int_{0}^{1} u\log u du = \frac{1}{2} < \infty.$$

Hence M is a spectral function.

Let Q be an infinitely divisible measure for which M is the corresponding spectral function. Since

$$(-1)^{i+1}\frac{d\overline{M_i^s}(\bar{u})}{d\bar{u}} = \overline{N_i^s}(\bar{u}) \qquad (i = 1, 2),$$

we have $D^{sd}(Q) = D^{id}(P)$. Thus $D^{sd}(Q) = S$. This completes the proof.

REFERENCES

[1] Ю. В. Линник и И. В. Островский, *Разложения случайных величин и векторов*, Москва 1972.
[2] M. Loève, *Probability theory*, New York 1950.
[3] T. Niedbalska, *An example of the decomposability semigroup*, Colloquium Mathematicum 39 (1978), p. 137-139.
[4] K. Urbanik, *Operator semigroups associated with probability measures*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 23 (1975), p. 75-76.
[5] — *Some examples of decomposability semigroups*, ibidem 24 (1976), p. 915-918.

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