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STRONGLY HOMOTOPICALLY STABILE POINTS

BY

A. LELEK (DETROIT, MICHIGAN)

The notion of a point of stability of a function plays a role in dimension theory and its applications to embeddings of topological spaces into Euclidean spaces. For example, a well-known theorem says that a compact metric space is at least n-dimensional if and only if it can be transformed into the n-cube by a continuous mapping which is stabile at a point (see [7], p. 75-77). A modified version of this notion is applied in the present paper to curve theory and, in particular, to studying some local properties of arcwise connected continua. Among other things, a result is obtained on a stability of locally confluent mappings (see 3.1).

1. Definitions and examples. By a *mapping* we always mean a continuous function. Given two mappings $f\colon X \to Y$ and $g\colon X \to Y$, we write $f \simeq g$ to indicate that f and g are homotopic. We say that a mapping $f\colon X \to Y$ is *strongly homotopically stabile* at a point $x_0 \in X$ provided $f \simeq g$ implies $f(x_0) = g(x_0)$ for each mapping $g\colon X \to Y$. Equivalently, the mapping f is strongly homotopically stabile at x_0 provided, for each homotopy $h\colon X \times I \to Y$, where $I = [0, 1]$ and $h(x, 0) = f(x)$ for $x \in X$, we have $h(x_0, t) = f(x_0)$ for $t \in I$. Thus, if the identity mapping of a topological space X is strongly homotopically stabile at a point $x_0 \in X$, then the point x_0 is homotopically stabile in the sense investigated by Borsuk and Jaworowski [2], and originally due to Hopf and Pannwitz [6]. It follows directly from the definition that, for each mapping $f\colon X \to Y$ of a topological space X into a Hausdorff space Y, the set of all the points of X at which f is strongly homotopically stabile is closed in X. In the case where a Hausdorff space Y does not contain any arc containing a given point $y_0 \in Y$, it is clear that each mapping $f\colon X \to Y$ is strongly homotopically stabile at each point x_0 such that $f(x_0) = y_0$. In this paper we shall, however, be primarily concerned with spaces that contain many arcs.

A *continuum* is understood to mean a connected compact metric space, and one-dimensional continua are called *curves*. We say that a curve is *acyclic* provided each mapping of it into the circle is homotopic to a constant mapping. A *dendroid* is an arcwise connected acyclic curve. A mapping $f\colon X \to Y$ of a topological space X into a topological space Y is *interior* at a point $x_0 \in X$ provided $f(x_0) \in \operatorname{Int} f(U)$ for each open subset $U \subset X$ containing x_0 (see [12], p. 149). The mapping f is *open* if it is interior at each point of X.

1.1. Example. *There exist a dendroid D on the plane and a point $y_0 \in D$ such that each mapping f of a topological space X into D which is interior at a point $x_0 \in X$, where $f(x_0) = y_0$, is strongly homotopically stabile at x_0.*

Proof. We denote by $\overline{pq}$ the straight-line closed segment with end-points p and q. Taking the points

$$p = (-1, 0), \qquad q = (1, 0),$$
$$p_i = (0, -i^{-1}), \quad q_i = (0, i^{-1}) \quad (i = 1, 2, \ldots),$$

we define D to be the union

$$D = \overline{pq} \cup \bigcup_{i=1}^{\infty} (\overline{pp_i} \cup \overline{qq_i}).$$

Clearly, D is a dendroid. Let $y_0 = (0, 0)$. Assume that $f\colon X \to D$ is a mapping of a topological space X into D such that f is interior at $x_0 \in X$ with $f(x_0) = y_0$. Suppose, on the contrary, that f is not strongly homotopically stabile at x_0. It means that there exists a homotopy $h\colon X \times I \to D$ such that $h(x, 0) = f(x)$ for $x \in X$, and the set $h(\{x_0\} \times I)$ is non-degenerate. The latter set represents a path in D whose initial point is y_0. Since $p \neq y_0 \neq q$, there exists a number $t_0 \in I$ such that the set $h(\{x_0\} \times J)$, where $J = [0, t_0]$, is a non-degenerate subset of $D \setminus \{p, q\}$. Thus $\{x_0\} \times J$ is contained in the set $h^{-1}(D \setminus \{p, q\})$ which is open in $X \times I$. Since J is compact, there exists an open subset $U \subset X$ such that $x_0 \in U$ and

$$U \times J \subset h^{-1}(D \setminus \{p, q\}). \tag{1}$$

The sets

$$D_1 = \overline{py_0} \cup \bigcup_{i=1}^{\infty} \overline{pp_i} \quad \text{and} \quad D_2 = \overline{qy_0} \cup \bigcup_{i=1}^{\infty} \overline{qq_i}$$

are subdendroids of D and their union is D. We prove that

$$h(\{x_0\} \times J) \subset D_j \quad (j = 1, 2). \tag{2}$$

If $h(\{x_0\} \times J) \not\subset D_j$, where $j = 1$ or 2, then there is a number $t_j \in J$ such that $h(x_0, t_j) \in D \setminus D_j$, so that (x_0, t_j) is a point of the open subset $h^{-1}(D \setminus D_j)$ of $X \times I$. Consequently, there must exist an open subset $V_j \subset X$ such that $x_0 \in V_j$ and

$$V_j \times \{t_j\} \subset h^{-1}(D \setminus D_j),$$

whence, by (1), h maps $(U\cap V_j)\times\{t_j\}$ into both $D\setminus D_j$ and $D\setminus\{p,q\}$. The set $U\cap V_j$ is open in X and it contains the point x_0. Since the mapping f is interior at x_0, we get

$$y_0 = f(x_0)\in \operatorname{Int} f(U\cap V_j),$$

which implies that the set $f(U\cap V_j)$ contains all but a finite number of the points p_i and q_i, as each of these sequences converges to the origin y_0. If $j=1$, take a point $p_{i_0}\in f(U\cap V_1)$. Then $p_{i_0}=f(x_1)$, where $x_1\in U\cap V_1$, and we have $h(x_1,t_1)\in D\setminus D_1$. Also, by (1),

$$h(\{x_1\}\times J)\subset D\setminus\{p\}.$$

Since

$$h(x_1,0) = f(x_1) = p_{i_0},$$

the set $h(\{x_1\}\times J)$ represents a path in $D\setminus\{p\}$ whose initial point is p_{i_0}. The point p, however, cuts the dendroid D into components such that the component of $D\setminus\{p\}$ containing p_{i_0} is $\overline{pp_{i_0}}\setminus\{p\}$, a subset of D_1. We obtain $h(\{x_1\}\times J)\subset D_1$ which contradicts the fact that $h(x_1,t_1)\notin D_1$. If $j=2$, the argument is completely analogous, with all the subscripts 1 changed to 2, and p,p_{i_0} replaced by q,q_{i_0}, respectively. Hence (2) is proved.

We notice that $D_1\cap D_2=\{y_0\}$. This, by (2), yields $h(\{x_0\}\times J)=\{y_0\}$. As a result, the set $h(\{x_0\}\times J)$ is degenerate, a contradiction completing the proof of 1.1.

1.2. Example. *There exist a dendroid E on the plane and a countable dense subset $B\subset E$ such that each mapping f of a topological space X into E which is interior at a point $x_0\in X$, where $f(x_0)\in B$, is strongly homotopically stabile at x_0.*

Consequently, if $f\colon X\to E$ is a mapping which is interior at each point of a dense subset $A\subset X$, where $f(A)\subset B$, then f is strongly homotopically stabile at each point of X; that is, $f\simeq g$ implies $f=g$ for each mapping $g\colon X\to E$. Hence, if $f\colon X\to E$ is an open mapping, then $f\simeq g$ implies $f=g$ for each mapping $g\colon X\to E$. In particular, the last implication holds for f being the identity mapping of E.

Proof. A construction of such a dendroid has been given in an earlier paper (see [5], p. 193). Here we only sketch the proof of its main property involving strong homotopic stability. The dendroid E is constructed by means of a sequence of countable collections $\boldsymbol{R}_n$ of some rhombi on the plane. We define B to be the set of the mid-points of longer diagonals of all the rhombi in $\boldsymbol{R}_n$ $(n=0,1,\ldots)$. Let $y_0\in B$ and let $R\in\boldsymbol{R}_n$ be the rhombus whose longer diagonal, say $\overline{pq}$, has y_0 as its mid-point. Those rhombi of $\boldsymbol{R}_{n+1}$ which are contained in R form the collection $\boldsymbol{R}(R)$ which is the union of four sequences of rhombi (ibidem). The first two of these sequences consist of rhombi attached to $\overline{pq}$ at the mid-point y_0. The third and the fourth sequences, however, consist of rhombi attached to $\overline{pq}$ at p and q, respectively, in the same manner as the segments $\overline{pp_i}$ and $\overline{qq_i}$ were attached to $\overline{pq}$ in the proof of 1.1. If $f\colon X \to E$ is a mapping of a topological space X into E such that f is interior at $x_0 \in X$ with $f(x_0) = y_0$, then the proof of 1.1 can be adapted, without any significant change, to this new situation. As a result, f is strongly homotopically stabile at x_0.

Remarks. The dendroid E from 1.2 is *connected im kleinen* at a point $y \in E$, that is, there exist arbitrarily small connected closed neighbourhoods of y in E. Actually, E is connected im kleinen at each point of a dense subset of E (cf. [5], p. 191). Nevertheless, E is not locally connected at any point (ibidem, p. 192). By a minor modification in the construction of E, one can, however, have another dendroid E' such that E' satisfies all the conditions listed in 1.2 and, in addition, E' is locally connected at each point of a dense subset of E'. Namely, one can strictly follow the pattern suggested by the construction of the dendroid D in 1.1, and instead of taking four sequences of rhombi whose union is $\boldsymbol{R}(R)$ in the definition of the dendroid (ibidem, p. 193) take only two of them: the third and the fourth ones. The new dendroid E' so obtained will be locally connected at each end-point of the longer diagonal of each rhombus that appears in the definition; the set of all such end-points will be dense in E'. On the other hand, it seems impossible here to achieve local connectedness everywhere. This is related to the following unsettled conjecture (cf. 2.6 below): for each locally connected continuum Y, there exists an open mapping $f\colon X \to Y$ of a continuum X onto Y such that f is not strongly homotopically stabile at any point of X (**P 1009**). According to a result of Anderson [1] (see also [9], Theorem 4.1), each locally connected continuum Y is the image of the Menger universal curve M under an open mapping f such that $f^{-1}(y)$ is homeomorphic to M for $y \in Y$. We do not know whether the Anderson mapping is strongly homotopically stabile at a point. If it were not, it would solve the above-mentioned conjecture. In this case, the fact that the components of the sets $f^{-1}(y)$ are non-degenerate would be essential (see 3.1).

2. Local acyclicity and confluency. We say that a curve X is *locally acyclic* at a point $x_0 \in X$ provided there exists a subset $A \subset X$ such that $x_0 \in \operatorname{Int} A$ and each mapping of A into the circle S is homotopic to a constant mapping. Clearly, the set of all the points of a curve X at which X is locally acyclic is open in X. The circle S itself is an example of a curve which is not acyclic although it is locally acyclic at each point of S.

2.1. *If X is a curve and $x_0 \in X$, then a necessary and sufficient condition for X to be locally acyclic at x_0 is that there exist a closed neighbourhood U of x_0 in X such that, for each closed subset $C \subset U$, each mapping of C into the circle is homotopic to a constant mapping.*

Proof. The condition is trivially sufficient. To see that it is also necessary, let us assume that X is locally acyclic at x_0, i.e., there is a set $A \subset X$ with $x_0 \in \operatorname{Int} A$ and each mapping $f\colon A \to S$ is homotopic to a constant mapping. Let $U \subset X$ be any closed subset such that $x_0 \in \operatorname{Int} U$ and $U \subset A$. If $C \subset U$ is a closed subset, then C is closed in A. Since $\dim A = 1$, each mapping $g\colon C \to S$ admits a continuous extension $f\colon A \to S$ (see [10], p. 354). Thus f is homotopic to a constant mapping, and so is $g = f|C$. The proof of 2.1 is completed.

We say that a mapping $f\colon X \to Y$ of a compact metric space X onto a compact metric space Y is *confluent* [3] provided, for each continuum $C \subset Y$ and each component K of $f^{-1}(C)$, we have $C = f(K)$. The mapping f is said to be *locally confluent* at a point $y_0 \in Y$ provided there exists a closed neighbourhood V of y_0 in Y such that $f|f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V (see [4], p. 239). When we say that a mapping f is locally confluent, without specifying at which point, it means that f is locally confluent at each point of its range. Obviously, each confluent mapping is locally confluent. All the surjective open mappings of compact metric spaces are confluent (see [12], p. 148). It is apparent that the class of confluent mappings also includes all those surjective mappings of compact metric spaces which are *monotone*, that is, have connected point-inverses.

2.2. THEOREM. *Let $f\colon X \to Y$ be a mapping of a compact metric space X into a curve Y and let $x_0 \in X$ be a point such that Y is not locally acyclic at $f(x_0)$ and the following condition is satisfied*:

($*$) *there exist arbitrarily small closed neighbourhoods U of x_0 in X for which $f(x_0) \in \operatorname{Int} f(U)$ and $f|U$ is a confluent mapping of U onto $f(U)$.*

Then f is strongly homotopically stabile at x_0.

Proof. Suppose, on the contrary, that f is not strongly homotopically stabile at x_0. It means that there exists a homotopy $h\colon X \times I \to Y$ such that $h(x, 0) = f(x)$ for $x \in X$ and $h(x_0, 1) \neq f(x_0)$. Write $f_1(x) = h(x, 1)$ for $x \in X$. Since $f(x_0) \neq f_1(x_0)$, we can find a closed neighbourhood U_0 of x_0 in X such that

$$f(U_0) \cap f_1(U_0) = \emptyset \tag{3}$$

and U_0 is one of the neighbourhoods whose existence is guaranteed by ($*$). Thus $f(x_0) \in \operatorname{Int} f(U_0)$ and $f|U_0$ is a confluent mapping of U_0 onto $f(U_0)$. But the curve Y is not locally acyclic at $f(x_0)$. By 2.1, there exist a closed subset $C \subset f(U_0)$ and a mapping $g\colon C \to S$ such that g is not homotopic to a constant mapping. We can assume that C is a continuum (see [10], p. 425 and 427). Since $f|U_0$ is confluent, there exists a component K of $(f|U_0)^{-1}(C)$ such that $C = f(K)$. Consequently, by (3) we have $C \cap f_1(K) = \emptyset$. Moreover, the mapping $f|K$ of K onto C is also confluent (see [3],

p. 213). Let

$$g_1\colon C \cup f_1(K) \to S$$

be the mapping defined by the formula

$$g_1(y) = \begin{cases} g(y) & \text{for } y \in C, \\ 1 & \text{for } y \in f_1(K). \end{cases}$$

Since Y is a curve, there exists a continuous extension $\bar{g}_1\colon Y \to S$ of g_1 (see [10], p. 354). Let us define a homotopy $h_1\colon K \times I \to S$ by means of the formula

$$h_1(x, t) = \bar{g}_1 h(x, t) \qquad (x \in K,\ t \in I).$$

For each point $x \in K$, we obtain

$$h_1(x, 0) = \bar{g}_1 h(x, 0) = \bar{g}_1 f(x) = gf(x),$$
$$h_1(x, 1) = \bar{g}_1 h(x, 1) = \bar{g}_1 f_1(x) = 1,$$

so that the composite $g\circ(f|K)$ is homotopic to a constant mapping. Since $f|K$ is confluent, it follows that g is homotopic to a constant mapping (see [11], p. 229), which is not the case; hence Theorem 2.2 is proved.

Remarks. If a mapping $f\colon X \to Y$ fulfills condition $(*)$ of 2.2 at a point $x_0 \in X$, then f is interior at x_0. Fulfilling condition $(*)$ at each point of X implies that f is an open mapping. There exist, however, open mappings which do not satisfy $(*)$ (see 2.3). A subclass of the class of open mappings has extensively been investigated. It is the class of those open mappings which are *light* [12], that is, have zero-dimensional point-inverses. The following question remains unanswered: is it true that each light open mapping of a compact metric space into a compact metric space satisfies condition $(*)$ at each point of its domain? (**P 1010**) We note that the conclusion of Theorem 2.2 holds for some light open mappings of compact metric spaces into curves (see 3.1). Example 2.3 indicates that the question would have been answered negatively if the lightness of the mapping had not been assumed.

2.3. Example. *There exist an irreducible continuum X on the plane and an open monotone mapping $f\colon X \to I$ such that f does not satisfy condition* $(*)$.

Proof. This example is due to Knaster [8] who constructed a continuum X on the plane such that X is irreducible between two points $p, q \in X$ and there is an open monotone mapping $f\colon X \to I$ with non-degenerate point-inverses, and with $f(p) = 0$ and $f(q) = 1$. Let $x_0 \in f^{-1}(\frac{1}{2})$. Since $f^{-1}(\frac{1}{2})$ is non-degenerate, there exists another point $x_1 \in f^{-1}(\frac{1}{2})$, $x_0 \neq x_1$. Suppose, on the contrary, that condition $(*)$ is satisfied. There exists then a closed neighbourhood U_0 of x_0 in X such that $x_1 \notin U_0$,

$f(x_0) \in \mathrm{Int} f(U_0)$, and $f|U_0$ is a confluent mapping of U_0 onto $f(U_0)$. Thus $f(x_0) = \frac{1}{2}$ and there exist numbers a and b such that

$$0 < a < \tfrac{1}{2} < b < 1, \qquad [a, b] \subset f(U_0).$$

Since $f|U_0$ is confluent, there exists a component K of the set $(f|U_0)^{-1}([a, b])$ with $[a, b] = f(K)$. Hence $K \subset U_0$ and $x_1 \notin K$. But $f(x_1) = \frac{1}{2}$, so that the set

$$K' = f^{-1}([0, a]) \cup K \cup f^{-1}([b, 1])$$

does not contain x_1. The points a and b are in $f(K)$, whence K meets both $f^{-1}([0, a])$ and $f^{-1}([b, 1])$. It follows that K' is a subcontinuum of X which joins p and q. This contradicts the fact that the continuum X is irreducible between p and q.

2.4. *If X and Y are compact metric spaces and $f: X \times Y \to Y$ is the projection of the product $X \times Y$ onto Y, then f satisfies condition $(*)$ of 2.2 at each point of $X \times Y$.*

Proof. Let $(x_0, y_0) \in X \times Y$ and let U and V be any closed neighbourhoods of x_0 and y_0 in X and Y, respectively. Then $U \times V$ is an (arbitrarily small) closed neighbourhood of (x_0, y_0) in $X \times Y$, and $y_0 = f(x_0, y_0)$ is an interior point of $V = f(U \times V)$. Let $C \subset V$ be any continuum and let K be a component of

$$U \times C = (f|U \times V)^{-1}(C).$$

Thus $f(K) \subset C$. Let $(x_1, y_1) \in K$ be any point. Since $x_1 \in U$, we have $\{x_1\} \times C \subset U \times C$, where $\{x_1\} \times C$ is a continuum. The latter continuum meets K, as $y_1 \in C$. It follows that $\{x_1\} \times C \subset K$, whence

$$C = f(\{x_1\} \times C) \subset f(K)$$

and, therefore, $C = f(K)$. But $f(K) = (f|U \times V)(K)$, so that $f|U \times V$ is, indeed, a confluent mapping of $U \times V$ onto V, and condition $(*)$ is satisfied. This completes the proof of 2.4.

2.5. COROLLARY. *If X is a compact metric space and Y is a curve such that Y is not locally acyclic at any point, then the projection of $X \times Y$ onto Y is strongly homotopically stabile at each point of $X \times Y$. In particular, the identity mapping of Y is strongly homotopically stabile at each point of Y.*

2.6. *If Y is a topological space, $A \subset Y$ is an arc, and $y_0 \in A$, then there exist a topological space X, a monotone mapping $f: X \to Y$, and a point $x_0 \in X$ such that $f(x_0) = y_0$ and f is not strongly homotopically stabile at x_0.*

Proof. Let $A' \subset A$ be a subarc one of end-points of which is y_0. Let $g: I \to A'$ be a homeomorphism of I onto A' such that $g(0) = y_0$. We define

$$X = (Y \times \{0\}) \cup (\{y_0\} \times I), \qquad x_0 = (y_0, 1),$$

and $f(x) = h(x, 0)$ $(x \in X)$, where the homotopy $h\colon X \times I \to Y$ is given by the formula

$$h\big((y, s), t\big) = \begin{cases} y & \text{for } (y, s) \in Y \times \{0\},\ t \in I, \\ g(st) & \text{for } (y, s) \in \{y_0\} \times I,\ t \in I. \end{cases}$$

Remarks. The product projections constitute a special case of open mappings. Since the Menger universal curve and the Sierpiński planar universal curve (see [10], p. 275) are not locally acyclic at any point, it follows from 2.5 that the product projections onto these curves are strongly homotopically stabile at each point. Some other mappings onto these curves, including light open mappings, have also the same property (see 3.2 and 3.3). However, by 2.6, monotone mappings onto either the Menger curve or the Sierpiński curve need not be strongly homotopically stabile. These two curves are locally connected without being locally acyclic. The dendroid E of 1.2 is an example of a curve which is acyclic without being locally connected and also has the property that the product projections onto E are strongly homotopically stabile at each point. As indicated in the Remarks following 1.2, the dendroid E can be modified so that the new dendroid E' becomes locally connected at each point of a dense subset. We point out that a stronger combination of local connectedness and local acyclicity leads to a different result (see 2.7).

Given a compact metric space X, we denote by $K(X)$ the set of all the points of X at which X is connected im kleinen. It is rather obvious that $K(X)$ is a G_δ-subset of X and that X is locally connected at each point of $\operatorname{Int} K(X)$ (see [5], p. 189).

2.7. THEOREM. *If $f\colon X \to Y$ is a mapping of a metric space X into a curve Y and $x_0 \in X$ is a point such that $f(x_0) \in \operatorname{Int} K(Y)$, and Y is locally acyclic at $f(x_0)$, then f is not strongly homotopically stabile at x_0.*

Proof. Write $y_0 = f(x_0)$. Since the curve Y is locally acyclic at y_0, there exists, by 2.1, a closed neighbourhood V_0 of y_0 in Y such that, for each closed subset $C \subset V_0$, each mapping of C into the circle is homotopic to a constant mapping. The set

$$G_0 = [\operatorname{Int} K(Y)] \cap (\operatorname{Int} V_0) \tag{4}$$

is an open neighbourhood of y_0 in Y. Since Y is connected im kleinen at y_0, there exists a continuum $C_0 \subset G_0$ such that $y_0 \in \operatorname{Int} C_0$.

We claim that the continuum C_0 is locally connected. Suppose that it is not. Then C_0 contains a non-degenerate continuum C' of convergence (see [10], p. 245). It means that

$$C' = \operatorname*{Lim}_{i\to\infty} C_i, \quad C' \cap C_i = \emptyset \quad (i = 1, 2, \ldots),$$

where C_i are subcontinua of C_0. Let $p, q \in C'$ be two distinct points. Since $C' \subset C_0 \subset G_0 \subset K(Y)$, by (4), the continuum Y is connected im kleinen at p and q. Thus in Y there exist connected closed neighbourhoods U_1 and U_2 of p and q, respectively, such that $U_1, U_2 \subset G_0$ and $U_1 \cap U_2 = \emptyset$. Since C_i converge to C', there is a subscript i_0 with C_{i_0} intersecting both U_1 and U_2. The sets

$$K_1 = C' \cup U_1 \cup U_2 \quad \text{and} \quad K_2 = C_{i_0} \cup U_1 \cup U_2$$

are continua whose common part is $U_1 \cup U_2$, a non-connected set. Consequently, the continuum $K_1 \cup K_2$ which is a subset of G_0 is not unicoherent, and so it admits a mapping into the circle which is not homotopic to a constant mapping (ibidem, p. 437). Since $K_1 \cup K_2 \subset G_0 \subset V_0$, by (4), this contradicts the main property of V_0. We have proved that C_0 is locally connected.

By (4) we also have $C_0 \subset V_0$. Hence C_0 is a unicoherent locally connected subcontinuum of the curve Y. It follows that C_0 is a dendrite (ibidem, p. 442). Let us write $A = f^{-1}(C_0)$. Since $y_0 \in \operatorname{Int} C_0$, we get $x_0 \in \operatorname{Int} A$, i.e., the closure of $X \setminus A$ in X does not contain the point x_0. The union

$$B = (A \times \{0\}) \cup \big([A \cap \operatorname{cl}(X \setminus A)] \times I\big) \cup \{(x_0, 1)\}$$

is a closed subset of the product $A \times I$. Let $y_1 \in C_0$ be any point different from y_0. The formula

$$g(x, t) = \begin{cases} f(x) & \text{for } x \in A,\ t = 0, \\ f(x) & \text{for } x \in A \cap \operatorname{cl}(X \setminus A),\ t \in I, \\ y_1 & \text{for } x = x_0,\ t = 1, \end{cases}$$

defines a mapping $g\colon B \to C_0$. Since the dendrite C_0 is an AR-space (ibidem, p. 339), there exists a continuous extension $\bar{g}\colon A \times I \to C_0$ of g. A homotopy $h\colon X \times I \to Y$ can now be defined by the formula

$$h(x, t) = \begin{cases} \bar{g}(x, t) & \text{for } x \in A,\ t \in I, \\ f(x) & \text{for } x \in \operatorname{cl}(X \setminus A),\ t \in I, \end{cases}$$

whence

$$h(x, 0) = f(x) \quad \text{for } x \in X,$$

and

$$h(x_0, 1) = \bar{g}(x_0, 1) = g(x_0, 1) = y_1 \neq y_0.$$

As a result, the mapping f is not strongly homotopically stabile at x_0.

3. Light locally confluent mappings. The following theorem is an analogue of Theorem 2.2 for locally confluent mappings.

3.1. THEOREM. *Let $f\colon X \to Y$ be a mapping of a compact metric space X into a curve Y and let $x_0 \in X$ be a point such that Y is connected* *im kleinen at $f(x_0)$, Y is not locally acyclic at $f(x_0)$, f is locally confluent at $f(x_0)$, and*

$$(**) \qquad \dim_{x_0} f^{-1}f(x_0) = 0.$$

Then f is strongly homotopically stabile at x_0.

Proof. Put $y_0 = f(x_0)$. Since f is locally confluent at y_0, there exists a closed neighbourhood V of y_0 in Y such that $f|f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V. The connectedness im kleinen of Y at y_0 implies the existence of an infinite sequence of continua $C_i \subset Y$ such that

$$(5) \qquad \{y_0\} = \bigcap_{i=1}^{\infty} C_i, \; y_0 \in \operatorname{Int} C_i, \; C_{i+1} \subset C_i \subset V \qquad (i = 1, 2, \ldots).$$

For $i = 1, 2, \ldots$, let K_i be the component of $f^{-1}(C_i)$ which contains x_0. First, we need to prove that

$$(6) \qquad \lim_{i\to\infty} \operatorname{diam} K_i = 0.$$

The point x_0 belongs to each of the continua K_i. It follows that the set

$$K' = \operatorname{Ls}_{i\to\infty} K_i$$

is a continuum (see [10], p. 171). Also, by (5) we have

$$K' = \bigcap_{i=1}^{\infty} \operatorname{cl} \bigcup_{j=0}^{\infty} K_{i+j} \subset \bigcap_{i=1}^{\infty} \operatorname{cl} f^{-1}(C_i) = f^{-1}\Big(\bigcap_{i=1}^{\infty} C_i\Big) = f^{-1}(y_0).$$

If (6) were not true, K' would be a non-degenerate continuum contained in $f^{-1}(y_0)$ and containing x_0, contrary to $(**)$.

To complete the proof of Theorem 3.1, let us suppose that the conclusion of this theorem does not hold, i.e., there exists a mapping $f_1\colon X \to Y$ such that $f \simeq f_1$ and $f(x_0) \neq f_1(x_0)$. Then there is a neighbourhood U of x_0 in X such that $f(U) \cap f_1(U) = \emptyset$. By (6), there exists a positive integer i_0 with $K_{i_0} \subset U$. Since $C_{i_0} \subset V$ by (5), and $f|f^{-1}(V)$ is a confluent mapping of $f^{-1}(V)$ onto V, we have $C_{i_0} = f(K_{i_0})$, and $f|K_{i_0}$ is a confluent mapping of K_{i_0} onto C_{i_0} (see [3], p. 213). Thus

$$C_{i_0} \cap f_1(K_{i_0}) = \emptyset.$$

On the other hand, since Y is not locally acyclic at y_0 and $y_0 \in \operatorname{Int} C_{i_0}$, a mapping of C_{i_0} into the circle is not homotopic to a constant mapping. We have seen in the proof of Theorem 2.2 that the existence of such continua $C = C_{i_0}$ and $K = K_{i_0}$ leads to a contradiction.

3.2. Corollary. *Each light locally confluent mapping of a compact metric space X onto a locally connected curve which is not locally acyclic at any point is strongly homotopically stabile at each point of X.*

3.3. COROLLARY. *If $f: X \to Y$ is a light open mapping of a compact metric space X onto a locally connected curve Y which is not locally acyclic at any point, then $f \simeq g$ implies $f = g$ for each mapping $g: X \to Y$.*

Remarks. According to 2.6, there exists a monotone mapping f of a curve onto the Sierpiński curve such that f is not strongly homotopically stabile. Then f is confluent, hence locally confluent. Consequently, the zero-dimensionality of point-inverses is a necessary condition in both 3.1 and 3.2. As condition (**) of 3.1 suggests, there may exist a higher dimension analogue of Theorem 3.1. To find such an analogue is, however, an open problem (**P 1011**). This problem seems to be related to an earlier one, namely to the problem of finding an appropriate definition of an n-dimensional analogue of confluent mappings such that the homotopy cancellation rule holds for mappings into the n-sphere (see [11], p. 233). When such a definition is found, one could hope to be able to generalize Theorem 3.1 to cover some types of mappings of compact metric spaces into n-dimensional continua ($n \geqslant 2$).

REFERENCES

[1] R. D. Anderson, *Open mappings of compact continua*, Proceedings of the National Academy of Sciences of the United States of America 42 (1956), p. 347-349.

[2] K. Borsuk and J. W. Jaworowski, *On labil and stabil points*, Fundamenta Mathematicae 39 (1952), p. 159-175.

[3] J. J. Charatonik, *Confluent mappings and unicoherence of continua*, ibidem 56 (1964), p. 213-220.

[4] R. Engelking and A. Lelek, *Metrizability and weight of inverses under confluent mappings*, Colloquium Mathematicum 21 (1970), p. 239-246.

[5] B. Fitzpatrick, Jr., and A. Lelek, *Some local properties of Suslinian compacta*, ibidem 31 (1974), p. 189-197.

[6] H. Hopf and E. Pannwitz, *Über stetige Deformationen von Komplexen in sich*, Mathematische Annalen 108 (1933), p. 433-465.

[7] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press 1948.

[8] B. Knaster, *Un continu irréductible à décomposition continue en tranches*, Fundamenta Mathematicae 25 (1935), p. 568-577.

[9] W. Kuperberg and A. Lelek, *Homotopy properties of pathwise connected continua*, ibidem 92 (1976), p. 29-41.

[10] K. Kuratowski, *Topology*, Vol. II, Academic Press 1968.

[11] A. Lelek, *On confluent mappings*, Colloquium Mathematicum 15 (1966), p. 223-233.

[12] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Publications 1942.

UNIVERSITY OF HOUSTON
WAYNE STATE UNIVERSITY

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