COLLOQUIUM MATHEMATICUM

VOL. XLVI 1982 FASC. 2

ON THE FORMULA OF ŚLEBODZIŃSKI FOR LIE DERIVATIVE OF TENSOR FIELDS IN A DIFFERENTIAL SPACE

BY

HANNA MATUSZCZYK (WROCŁAW)

In this paper we introduce the concept of a fibered tensor space over a differential space in a way such that each smooth tensor field is a smooth section of that tensor space. Lie derivative with respect to a vector field is introduced by means of the one-parameter Lie group, and the fundamental Ślebodziński's formula is derived.

1. Tangent and cotangent fibered space. Let (M, C) be a differential space and let τ_C stand for the weakest topology on M such that all $\alpha \in C$ are continuous, whereas C_U stands for the set of all $\beta\colon U \to \mathbf{R}$ such that for each $p \in U$ there exist $V \in \tau_C$ and $\alpha \in C$ with the condition $\beta|V = \alpha|V$; $p \in V$ (see [1]). For any $p \in M$ we define the set $C(p)$ as the union of all sets C_U, where $p \in U \in \tau_C$. The tangent space $(M, C)_p$, by Sikorski's definition (cf. [2] and [3]), is isomorphic to the vector space $T_p(M, C)$ of all mappings $v\colon C(p) \to \mathbf{R}$ such that

$$v(\alpha+\beta) = v(\alpha)+v(\beta), \quad v(c\alpha) = cv(\alpha),$$
$$v(\alpha\beta) = \alpha(p)v(\beta)+\beta(p)v(\alpha)$$

for α, $\beta \in C(p)$ and $c \in \mathbf{R}$. Here $\alpha+\beta$ and $\alpha\beta$ are defined on the common part of the domains of α and β and, of course, belong to $C(p)$ whenever $\alpha, \beta \in C(p)$.

PROPOSITION 1.1. *If (M, τ_C) is a Hausdorff space, then for any p, q in M, $p \neq q$, the tangent spaces $T_p(M, C)$ and $T_q(M, C)$ are disjoint.*

Proof. For $p, q \in M$, $p \neq q$, there exists $U \in \tau_C$ such that $p \in U$ and $q \notin U$. Then, for some $\alpha \in C$, we have $\alpha(p) \neq \alpha(q)$. Hence $\alpha|U \in C(p)$ and $\alpha|U \notin C(q)$. In other words, $C(p) \neq C(q)$. Thus $T_p(M, C)$ and $T_q(M, C)$ are disjoint.

For pure technical reasons we will assume that (M, τ_C) is a Hausdorff space.

It follows from Proposition 1.1 that we have exactly one mapping π, defined on the union of all vector spaces $T_p(M, C)$, $p \in M$, which sends every element v of this union to the point $p \in M$ such that v belongs to $T_p(M, C)$. This mapping is called the *projection of the tangent fibered space* of (M, C). In the union of all vector spaces $T_p(M, C)$, $p \in M$, we take the smallest differential structure containing the set $\{a \circ \pi;\ a \in C\} \cup \{a_*;\ a \in C\}$, where $a_*(v) = v(a)$ for v of the domain of π. The domain of π together with that differential structure will be called the *tangent bundle* of the differential space (M, C) and will be denoted by $T(M, C)$.

Any mapping X which assigns to each point $p \in M$ the vector $X(p)$ of $T_p(M, C)$ is called a *vector field* on (M, C). The vector field X is said to be *smooth* if for every $a \in C$ the function $\partial_X a$ defined by the formula

$$(\partial_X a)(p) = X(p)(a) \quad \text{for } p \in M$$

belongs to C. Let $\mathscr{X}(M, C)$ denote the set of all smooth vector fields on (M, C). By an easy verification we get

PROPOSITION 1.2. *Any vector field X on (M, C) is smooth if and only if $X\colon (M, C) \to T(M, C)$ is a smooth mapping.*

Now we define the cotangent bundle of a Hausdorff differential space. Any element of the space $(T_p(M, C))^*$, i.e. of the dual vector space to $T_p(M, C)$, is said to be a *tangent covector of (M, C) at the point p*. A tangent covector of (M, C) at any point $p \in M$ will be called, shortly, a *tangent covector of (M, C)*. It follows from Proposition 1.1 that for every tangent covector w of (M, C) there exists exactly one point $^*\pi(w)$ such that w is a tangent covector of (M, C) at $^*\pi(w)$. So we have a mapping $^*\pi$ of the set of all covectors of (M, C) onto M. In what follows we shall write π instead of $^*\pi$. Every covector w of (M, C) is an element of $(T_{\pi(w)}(M, C))^*$.

Let X be a smooth vector field on (M, C). For any tangent covector w of (M, C) we set $\tilde{X}(w) = w\big(X(\pi(w))\big)$. It is easy to see that we have the real-valued function $\tilde{X}$ linear on each space $(T_p(M, C))^*$. Taking the smallest differential structure on the set of all tangent covectors of (M, C) containing the set $\{a \circ \pi;\ a \in C\} \cup \{\tilde{X};\ X \in \mathscr{X}(M, C)\}$, we obtain a differential space which will be called the *cotangent fibered space* of (M, C) and denoted by $T^*(M, C)$. We have also the smooth map $\pi\colon T^*(M, C) \to (M, C)$ called the *projection of the cotangent bundle* of (M, C).

2. Tensor fibered space. For any Hausdorff differential space (M, C) and for any point $p \in M$ we consider the tensor algebra $\otimes T_p(M, C)$ over the linear space $T_p(M, C)$. We have the natural inclusions of $T_p(M, C)$ and $(T_p(M, C))^*$ into $\otimes T_p(M, C)$. In this section we shall write $T_p^1(M, C)$ instead of $T_p(M, C)$, and $T_p^{-1}(M, C)$ instead of $(T_p(M, C))^*$. For any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, where $\varepsilon_h \in \{-1, 1\}$, we denote by $\otimes^\varepsilon T_p(M, C)$ the linear subspace of the linear space $\otimes T_p(M, C)$ generated by the set of all tensors $v_1 \otimes \ldots \otimes v_k$, where v_h is from $T_p^{\varepsilon_h}(M, C)$, $h = 1, \ldots, k$. Any element of $\otimes^\varepsilon T_p(M, C)$ is said to be a *tensor of type* ε at the point p. A mapping which assigns to every point $p \in M$ a tensor $Z(p)$ of type ε at p is said to be a *tensor field of type* ε on (M, C). As above, denoting by π the mapping which assigns to every element t of the union of all sets of $\otimes T_p(M, C)$, $p \in M$, the point $\pi(t)$ such that t is from $\otimes T_{\pi(t)}(M, C)$, we may regard Z as a mapping such that $\pi \circ Z = \mathrm{id}_M$.

Before defining the differential structure on the set of all tensors on (M, C) we consider basic smooth functions γ defined as follows. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, $\varepsilon_h \in \{-1, 1\}$. Let $\operatorname{set}\otimes^{\varepsilon} T(M, C)$ be the union of all $\otimes^{\varepsilon} T_p(M, C)$, where $p \in M$. Let $\gamma_h = a_{h*}$, $a_h \in C$, if $\varepsilon_h = 1$, and $\gamma_h = \tilde{X}_h$, $X_h \in \mathscr{X}(M, C)$, if $\varepsilon_h = -1$. There exists exactly one function $[\gamma_1, \ldots, \gamma_h]$ defined on $\operatorname{set}\otimes^{\varepsilon} T(M, C)$,

$$[\gamma_1, \ldots, \gamma_k](v_1 \otimes \ldots \otimes v_k) = \gamma_1(v_1) \ldots \gamma_k(v_k),$$

where $\pi(v_1) = \ldots = \pi(v_k)$. We can extend this function to the function γ defined on $\operatorname{set}\otimes T(M, C)$, the union of all sets of elements of $\otimes T_p(M, C)$, where $p \in M$, by setting $\gamma(t) = [\gamma_1, \ldots, \gamma_k](t)$ for $t \in \operatorname{set}\otimes^{\varepsilon} T(M, C)$ and $\gamma(t) = 0$ for $t \notin \operatorname{set}\otimes^{\varepsilon} T(M, C)$.

Every tensor algebra $\otimes T_p(M, C)$ has the subalgebra of all scalars. This subalgebra, denoted by $\otimes^{(0)} T_p(M, C)$, is isomorphic in a natural way to the algebra of all real numbers. Similarly as before we denote the union of all sets of elements of $\otimes^{(0)} T_p(M, C)$, where $p \in M$, by $\operatorname{set}\otimes^{\varepsilon} T(M, C)$, $\varepsilon = (0)$. Every function γ defined on $\operatorname{set}\otimes T(M, C)$ by $\gamma(\hat{p}(r)) = r$ for any real r, where $\hat{p}$ is the natural isomorphism of all real numbers onto $\otimes^{(0)} T_p(M, C)$, and by $\gamma(t) = 0$ for $t \notin \otimes^{\varepsilon} T(M, C)$ will be also regarded as a basic smooth function.

The smallest differential structure on $\operatorname{set}\otimes T(M, C)$ containing all described basic smooth functions as well as all functions of the form $a \circ \pi$, where $a \in C$, is called the *differential structure of the tensor fibered space* $\otimes T(M, C)$ over (M, C). It is easy to prove the following

PROPOSITION 2.1. *If (M, C) is an m-dimensional differentiable manifold, then the differential space induced on* $\operatorname{set}\otimes^{\varepsilon} T(M, C)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, *by the differential structure of $\otimes T(M, C)$ is an $(m + m^k)$-dimensional differentiable manifold being the total space of the tensor bundle of all tensors of type ε, the base space of which is (M, C).*

3. One-parameter local Lie group. Let E denote the set of all real functions of class C^∞ on the set of all real numbers and let us consider any one-parameter local Lie group in a differential space (M, C), i.e. (cf. [5]) a smooth mapping

(1) $$\varphi\colon (U, C_U) \times (I, E_I) \to (M, C),$$

where $U \in \tau_C$, I is an open interval, $0 \in I$, satisfying the following conditions:

(i) if $t, s, t+s \in I$ and $p, \varphi(p, t) \in U$, then

$$\varphi(\varphi(p, t), s) = \varphi(p, t+s);$$

(ii) $\varphi(p, 0) = p$ for $p \in U$.

It follows immediately (cf. [5]) that for $t \in I$ we have the diffeomorphism

$$\varphi_t\colon (U_t, C_{U_t}) \to (U_{-t}, C_{U_{-t}}), \tag{2}$$

where $U_t = (\varphi(\cdot, t))^{-1}[U]$ and $\varphi_t(p) = \varphi(p, t)$ for $p \in U_t$, and for any $p \in U$ there exists $\delta > 0$ such that $p \in U_t$ for $t \in (-\delta, \delta)$. Of course, U_t is open for $t \in I$.

By $\dot{\varphi}(p, t)$ we denote the vector tangent to (M, C) at the point $\varphi(p, t)$ defined by the formula

$$\dot{\varphi}(p, t)(a) = (a \circ \varphi(p, \cdot))'(t) \quad \text{for } a \in C(p). \tag{3}$$

By an easy verification we get (cf. [5])

PROPOSITION 3.1. *For every one-parameter local Lie group* (1) *the formula*

$$X(p) = \dot{\varphi}(p, 0) \quad \text{for } p \in U \tag{4}$$

defines a smooth vector field X on U tangent to (M, C) and such that

$$X(\varphi(p, t)) = \dot{\varphi}(p, t) \quad \text{for } p \in U,\ t \in I.$$

The vector field defined by (3) is said to be *induced by the one-parameter local Lie group* (1).

PROPOSITION 3.2. *For any $p \in U$ there exists $\eta > 0$ such that for every $t \in (-\eta, \eta)$ we have the isomorphism*

$$\varphi_{t*p_t}\colon T_{p_t}(U_t, C_{U_t}) \to T_p(U_{-t}, C_{U_{-t}}), \tag{5}$$

where $p_t = \varphi(p, -t) \in U$. This isomorphism induces in a natural way the following three isomorphisms:

$$\varphi_t^{(1)}(p)\colon T_{p_t}(M, C) \to T_p(M, C), \tag{6}$$

$$\varphi_t^{(-1)}(p)\colon (T_{p_t}(M, C))^* \to (T_p(M, C))^*, \tag{7}$$

$$\tilde{\varphi}_t(p)\colon \otimes T_{p_t}(M, C) \to \otimes T_p(M, C). \tag{8}$$

Moreover, for any v_h in $T_{p_t}^{\varepsilon_h}(M, C)$, $h = 1, \ldots, k$, regarded as elements of $\otimes T_{p_t}(M, C)$, we have

$$\tilde{\varphi}_t(p)(v_1 \otimes \ldots \otimes v_k) = \varphi_t^{(\varepsilon_1)}(p)(v_1) \otimes \ldots \otimes \varphi_t^{(\varepsilon_k)}(p)(v_k), \tag{9}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$.

A system

$$W_1, \ldots, W_m \tag{10}$$

of smooth vector fields on a differential space (M, C) is said to be a *local vector basis* at the point p (cf. [2]) if there exists a neighbourhood U of p such that for any $q \in U$ the vectors $W_1(q), \ldots, W_m(q)$ are a basis for the vector space $T_q(M, C)$. The differential space (M, C) is said to be of *local dimension* m at the point p if there exists a local vector basis (10) at p. The vector fields W_h can be defined only on some neighbourhood of p. By C-regularity of (M, C), this is equivalent to our assumption.

PROPOSITION 3.3. *If (M, C) is of local dimension m at the point p, then for any tensor field Z of type $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, smooth on (M, C), and for any $i_1, \ldots, i_k \in \{1, \ldots, m\}$ there are functions $Z_{i_1 \ldots i_k}$ of class C^∞ on $(-\eta, \eta)$ such that for $t \in (-\eta, \eta)$ we have*

$$Z(p_t) = \sum_{i_1, \ldots, i_k} Z_{i_1 \ldots i_k}(t) e_{i_1}^{\varepsilon_1}(t) \otimes \ldots \otimes e_{i_k}^{\varepsilon_k}(t), \tag{11}$$

where $p_t = \varphi(p, -t)$, $e_1^1(t), \ldots, e_m^1(t)$ is a basis for the vector space $T_{p_t}(M, C)$ and $e_1^{-1}(t), \ldots, e_m^{-1}(t)$ is its dual basis. Moreover,

$$\varphi_t^{(\mu)}\big(e_h^\mu(t)\big) = e_h^\mu(0), \quad h = 1, \ldots, m, \ |\mu| = 1. \tag{12}$$

Proof. Let $W_1, \ldots, W_m$ be a local vector basis at p. From C-regularity of (M, C) (see [3]) it follows that there exist smooth 1-forms $W^1, \ldots, W^m$ defined on (M, C), i.e. some mappings $W^i\colon (M, C) \to \otimes T(M, C)$ are smooth tensor fields of type (-1) such that $W^i(q)\big(W_h(q)\big) = \delta_h^i$ for q in some neighbourhood of the point p, and $h, i = 1, \ldots, m$. Proposition 3.1 yields the existence $\eta > 0$ such that for $t \in (-\eta, \eta)$ we have the isomorphisms (5)-(8).

After identification of $T_q(M, C)$ with $\otimes^{(1)} T_q(M, C)$ and $\big(T_q(M, C)\big)^*$ with $\otimes^{(-1)} T_q(M, C)$ we set, for $t \in (-\eta, \eta)$,

$$e_h^1(t) = \big(\varphi_t^{(1)}(p)\big)^{-1}\big(W_h(p)\big), \quad h = 1, \ldots, m, \tag{13}$$

$$e_i^{-1}(t) = \big(\varphi_t^{(-1)}(p)\big)^{-1}\big(W^i(p)\big), \quad i = 1, \ldots, m. \tag{14}$$

It is easy to check that we have the basis $e_1^1(t), \ldots, e_m^1(t)$ for $\otimes^{(1)} T_{p_t}(M, C)$ and its dual $e_1^{-1}(t), \ldots, e_m^{-1}(t)$ for $\otimes^{(-1)} T_{p_t}(M, C)$. The elements $e_h^1(t)$ and $e_h^{-1}(t)$ are identified with $\varphi(\cdot, -t)_{*p}\big(W_h(p)\big)$ and $W^h(p) \circ \varphi(\cdot, t)_{*p_t}$, respectively. It is easy to verify that $e_h^1(0)$ equals $W_h(p)$ regarded as an element of $\otimes^{(1)} T_p(M, C)$. Similarly, $e_h^{-1}(0)$ is identified with $W^h(p)$. Hence (12) holds.

The vectors $W_1(p), \ldots, W_m(p)$ are linearly independent. Then there exist functions $a^1, \ldots, a^m \in C$ such that

$$\det[W_h(p)(a^j);\ h, j \leqslant m] \neq 0.$$

Let us remark that there exist real functions a_0^j belonging to the differential structure of the space $(U, C_U)\times(I, E_I)$, i.e. of the domain of the mapping (1), such that

$$a^j(\varphi(q, -t)) = a^j(q)+ta_0^j(q, t) \quad \text{for } q\in U,\ t\in I.$$

Hence

$$\varphi(\cdot, -t)_{*p}(W_h(p))(a^j) = W_h(p)(a^j\circ\varphi(\cdot, -t)) = W_h(p)(a^j)+tW_h(p)(a_0^j(\cdot, t)).$$

Therefore, diminishing η (if necessary), we may assume that

$$\det[\varphi(\cdot, -t)_{*p}(W_h(p))(a^j);\ h, j\leqslant m]\neq 0 \quad \text{for } t\in(-\eta, \eta). \tag{15}$$

Then there exist real functions β_j^i of class C^∞ on $(-\eta, \eta)$ such that for $t\in(-\eta, \eta)$ we have

$$\beta_j^i(t)\varphi(\cdot, -t)_{*p}(W_h(p))(a^j) = \delta_h^i, \quad h, i = 1, \ldots, m. \tag{16}$$

Let θ_h^l be real functions on $(-\eta, \eta)$ such that

$$\varphi(\cdot, t)_{*p_t}(W_h(\varphi(p, -t))) = \theta_h^l(t)W_l(p). \tag{17}$$

Thus we get

$$W_h(\varphi(p, -t)) = \theta_h^l(t)\varphi(\cdot, -t)_{*p}(W_l(p)).$$

Hence, by (16), we obtain

$$\beta_j^i(t)W_h(\varphi(p, -t))(a^j) = \theta_h^l(t)\beta_j^i(t)\varphi(\cdot, -t)_{*p}(W_l(p))(a^j) = \theta_h^l(t)\delta_l^i = \theta_h^i(t).$$

Then θ_h^i are of class C^∞ on $(-\eta, \eta)$. Applying $W^i(p)$ to both sides of (17) we obtain

$$\theta_h^i(t) = W^i(p)(\varphi(\cdot, t)_{*p_t})(W_h(\varphi(p, -t))), \quad h,\ i = 1, \ldots, m. \tag{18}$$

On the other hand, by the definition of the operation $X\mapsto\tilde{X}$, we have

$$\tilde{W}_h(W^i(p)\circ\varphi(\cdot, t)_{*p_t}) = (W^i(p)\circ\varphi(\cdot, t)_{*p_t})(W_h(\varphi(p, -t))). \tag{19}$$

It follows from (14) that $e_i^{-1}(t)$ can be identified with the covector $W^i(p)\circ\varphi(\cdot, t)_{*p_t}(W_h(\varphi(p, -t)))$. Setting for v and w in $\otimes T(M, C)$,

$$\gamma_h^{(1)}(v) = \begin{cases} a_*^h(v) & \text{if } v \text{ is in } \otimes^{(1)}T(M, C),\\ 0 & \text{otherwise,}\end{cases} \tag{20}$$

and

$$\gamma_h^{(-1)}(v) = \begin{cases} \tilde{W}_h(w) & \text{if } w \text{ is in } \otimes^{(-1)}T(M, C),\\ 0 & \text{otherwise,}\end{cases} \tag{21}$$

by (11) we obtain

$$[\gamma_{h_1}^{(\varepsilon_1)}, \ldots, \gamma_{h_k}^{(\varepsilon_k)}](Z(p_t)) = \sum_{i_1,\ldots,i_k} Z_{i_1\ldots i_k}(t)\gamma_{h_1}^{(\varepsilon_1)}(e_{i_1}^{\varepsilon_1}(t))\ldots\gamma_{h_k}^{(\varepsilon_k)}(e_{i_k}^{\varepsilon_k}(t)). \tag{22}$$

According to (16) and (20) we have

$$\beta_j^i(t)\gamma_h^{(1)}\big(e_i^1(t)\big) = \beta_j^i(t)\alpha_*^h\big(\varphi(\cdot, -t)_{*p}\big(W_i(p)\big)\big) = \delta_j^h. \tag{23}$$

Similarly, by (19), (21), and (18) we get

$$\gamma_h^{(-1)}\big(e_i^{-1}(t)\big) = \tilde{W}_h\big(W^i(p)\circ\varphi(\cdot, t)_{*p_t}\big) = \theta_h^i(t). \tag{24}$$

From (23) and (24) it follows that

$$\gamma_h^{(\mu)}\big(e_i^\mu(t)\big) = \theta_{hi}^\mu(t), \tag{25}$$

where

$$\theta_{hi}^\mu = \begin{cases} \breve{\beta}_i^h & \text{if } \mu = 1, \\ \theta_h^i & \text{if } \mu = -1, \end{cases} \qquad \text{and} \qquad \breve{\beta}_i^h(t)\beta_h^j(t) = \delta_i^j.$$

Of course, $\det[\theta_{hi}^\mu(t);\ h, i \leqslant m] \neq 0$ for $t \in (-\eta, \eta)$. Hence and from (22) and (25), because of the smoothness of Z, it follows that the functions $Z_{i_1\ldots i_k}$ are of class C^∞ on $(-\eta, \eta)$, which proves the proposition.

4. Lie derivative of tensor fields. Let $T_q(M, C)$ be an m-dimensional space and $e_1, \ldots, e_m$ a basis for it. For any tensor u of type $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ and for r, s such that $1 \leqslant r < s \leqslant k$ and $\varepsilon_r\varepsilon_s = -1$ we denote by $C_{r,s}u$ the tensor of type $(\varepsilon_1, \ldots, \varepsilon_{r-1}, \varepsilon_{r+1}, \ldots, \varepsilon_{s-1}, \varepsilon_{s+1}, \ldots, \varepsilon_k)$ defined by the equality

$$C_{r,s}u = \sum_{\substack{i_1,\ldots,i_k \\ i_r = i_s}} u_{i_1\ldots i_k} e_{i_1}^{\varepsilon_1} \otimes \ldots \otimes e_{i_{r-1}}^{\varepsilon_{r-1}} \otimes e_{i_{r+1}}^{\varepsilon_{r+1}} \otimes \ldots \otimes e_{i_{s-1}}^{\varepsilon_{s-1}} \otimes e_{i_s+1}^{\varepsilon_{s+1}} \otimes \ldots \otimes e_{i_k}^{\varepsilon_k},$$

where

$$u = \sum_{i_1,\ldots,i_k} u_{i_1\ldots i_k} e_{i_1}^{\varepsilon_1} \otimes \ldots \otimes e_{i_k}^{\varepsilon_k}.$$

The operation $C_{r,s}$ is a contraction. In a particular case, taking v in $T_q(M, C)$ and w in $\big(T_q(M, C)\big)^*$ as elements of $\otimes T_q(M, C)$, after the contraction $C_{1,2}$ of the tensor product $w \otimes v$ we obtain

$$C_{1,2}(w \otimes v) = w(v). \tag{26}$$

Let W and Y be a smooth 1-form and a smooth vector field, respectively, on some open subspace of (M, C) being of local dimension m. Then, by (26), we have

$$C_{1,2}\circ(W \otimes Y) = W(Y), \tag{27}$$

where $W(Y)(q) = W(q)\big(Y(q)\big)$ for q of this subspace.

Let us consider a differential space (M, C) being of local dimension m at the point p and a one-parameter local Lie group (1) such that $p \in U$. For sufficiently small t, by Proposition 3.3, we have formulas (11) and (12).

Hence

$$\tilde{\varphi}_t(p)\big(Z(p_t)\big) = \sum_{i_1,\ldots,i_k} Z_{i_1\ldots i_k}(t)e_{i_1}^{\varepsilon_1}(0)\otimes\ldots\otimes e_{i_k}^{\varepsilon_k}(0). \tag{28}$$

It is easy to check that the derivative of the function $t\mapsto\tilde{\varphi}_t(p)\big(Z(p_t)\big)$ at the point 0 is independent of the choice of a local basis $W_1,\ldots,W_m$. Therefore, we can denote it by $-(\pounds_\varphi Z)(p)$.

PROPOSITION 4.1. *If (M, C) is of local dimension m at p, then the operation $\pounds_\varphi$ is linear, commutes with all contractions, and for any smooth tensor fields Z and Z_1 of type ε and η, respectively, we have*

$$\big(\pounds_\varphi(Z\otimes Z_1)\big)(p) = (\pounds_\varphi Z)(p)\otimes Z_1(p)+Z(p)\otimes(\pounds_\varphi Z_1)(p). \tag{29}$$

Moreover, if W and Y are a smooth 1-form and a smooth vector field, respectively, then

$$(\pounds_\varphi W)\big(Y(p)\big) = \big(\partial_X W(Y)\big)(p)-W([X, Y])(p), \tag{30}$$

where X is the vector field induced by φ.

Proof. It follows from (28) that

$$(\pounds_\varphi Z)(p) = -\sum_{i_1,\ldots,i_k} Z'_{i_1\ldots i_k}(0)e_{i_1}^{\varepsilon_1}(0)\otimes\ldots\otimes e_{i_k}^{\varepsilon_k}(0). \tag{31}$$

Since the mapping (8) is linear and commutative with contractions, so is $\pounds_\varphi$. Formula (29) is also an immediate consequence of (31). Formula (30) follows from (27), (29), and from the equality $(\pounds_\varphi Y)(p) = [X, Y](p)$ (see [5]), which completes the proof.

It follows from Proposition 4.1, because of the local character of the operation $\pounds_\varphi$, that $\pounds_\varphi$ depends only on the vector field X induced by φ. Thus we may define Lie derivative $(\pounds_X Z)(p)$ of the tensor field Z with respect to the vector field X at the point p as the tensor $(\pounds_\varphi Z)(p)$ considered above. Indeed, for q in some neighbourhood of p we have

$$Z(q) = \sum_{i_1,\ldots,i_k} a_{i_1\ldots i_k}(q)\, W_{i_1}^{\varepsilon_1}(q)\otimes\ldots\otimes W_{i_k}^{\varepsilon_k}(q), \tag{32}$$

where $W_h^1(q)$ stands for $W_h(q)$ regarded as an element of $\otimes T_q(M, C)$, and $W_i^{-1}(q)$ is an element of $\otimes T_q(M, C)$ corresponding to $W^i(q)$, $W^i(q)\big(W_h(q)\big) = \delta_h^i$, $h, i = 1,\ldots,m$. Applying the operation $\pounds_\varphi$ to (32) we get

$$\begin{aligned}(\pounds_\varphi Z)(p) = &\sum_{i_1,\ldots,i_k} (\pounds_\varphi a_{i_1\ldots i_k})(p)W_i^{\varepsilon_1}(p)\otimes\ldots\otimes W_{i_k}^{\varepsilon_k}(p)+\\ &+\sum_{i_1,\ldots,i_k} a_{i_1\ldots i_k}(p)\sum_{r=1}^{k} W_{i_1}^{\varepsilon_1}(p)\otimes\ldots\otimes(\pounds_\varphi W_{i_r}^{\varepsilon_r})(p)\otimes\ldots\otimes W_{i_k}^{\varepsilon_k}(p).\end{aligned}$$

Hence

$$(33)\qquad (\pounds_X Z)(p) = \sum_{i_1,\ldots,i_k} X(p)(a_{i_1\ldots i_k}) W_{i_1}^{\varepsilon_1}(p)\otimes \ldots \otimes W_{i_k}^{\varepsilon_k}(p) +$$
$$+\sum_{i_1,\ldots,i_k} a_{i_1\ldots i_k}(p)\sum_{r=1}^{k} W_{i_1}^{\varepsilon_1}(p)\otimes \ldots \otimes (\pounds_X W_{i_r}^{\varepsilon_r})(p)\otimes \ldots \otimes W_{i_k}^{\varepsilon_k}(p),$$

where

$$(34)\qquad (\pounds_X W_h^{\mu})(p) = [X, W_h](p)$$

if $\mu = 1$; and for any smooth vector field Y, $\mu = -1$, we get

$$(35)\qquad (\pounds_X W_i^{\mu})(Y(p)) = (\partial_X W^i(Y))(p) - W^i([X, Y])(p).$$

With these remarks, Proposition 4.1 yields

THEOREM 4.1. *If (M, C) is of local dimension m at the point p, then we have the operation $\pounds_X$ which assigns to every smooth tensor field Z of type ε the tensor $(\pounds_X Z)(p)$ defined as*

$$\lim_{t\to 0}\frac{1}{t}\Big(Z(p)-\tilde{\varphi}_t(p)\big(Z(\varphi(p,-t))\big)\Big),$$

where the one-parameter local Lie group (1) *induces X. The operation $\pounds_X$ is linear, commutes with all contractions, and satisfies the formula*

$$\pounds_X(Z\otimes Z_1)(p) = (\pounds_X Z)(p)\otimes Z_1(p) + Z(p)\otimes(\pounds_X Z_1)(p).$$

Moreover, formula (33) *together with* (34) *and* (35) *is satisfied.*

Formula (33) is a different formulation of the fundamental Ślebodziński's formula [4].

REFERENCES

[1] R. Sikorski, *Abstract covariant derivative*, Colloquium Mathematicum 18 (1967), p. 251-272.

[2] — *Differential modules*, ibidem 24 (1971), p. 45-79.

[3] — *Wstęp do geometrii różniczkowej* (*Introduction to differential geometry*), Warszawa 1972 (in Polish).

[4] W. Ślebodziński, *Sur les équations canoniques de Hamilton*, Bulletin de l'Académie Royale de Belgiques, Classe des Sciences, 5 Série, 17 (1931), p. 864-870.

[5] W. Waliszewski, *Lie derivative of vector fields on a differential space* (to appear).

Reçu par la Rédaction le 13. 7. 1980