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ON INTEGRAL TRANSFORMS OF HAAR FUNCTIONS

BY

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Let $\{h_n\}_{n=1}^{\infty}$ denote the normalized set of Haar functions, which form a normalized Schauder basis for $L^1[0, 1]$ (see [5], p. 13). It was observed by Ciesielski [1] that the set of functions $\{1, \int_0^t h_n(x)\,dx\}_{n=1}^{\infty}$ is a Schauder basis for $C[0, 1]$; in fact, these functions are simply (bounded) multiples of Schauder's original basis $\{\phi_n\}_{n=0}^{\infty}$ (see [5], p. 11) for $C[0, 1]$. Generalizing this result, Radecki [4] showed that if $\alpha(x)$ is any positive, continuous function of bounded variation on [0, 1], then the set $\{1, \int_0^t \alpha(x)\,h_n(x)\,dx\}_{n=1}^{\infty}$ is also a basis for $C[0, 1]$.

In this paper we prove a result of the same type, but where $\alpha(x)$ is assumed to be increasing on [0, 1] rather than continuous there (Theorem 2). Our proof not only shows that every such set of transforms of the Haar system is a basis for $C[0, 1]$, but that it is actually equivalent to the Schauder basis $\{\phi_n\}_{n=1}^{\infty}$. Using essentially the same techniques we then also show that the bases considered by Radecki (i.e. where $\alpha(x)$ is continuous, positive, and of bounded variation) are similarly equivalent to the basis $\{\phi_n\}_{n=0}^{\infty}$ (Theorem 3).

We begin with several simple lemmas. In what follows $C_0[0, 1]$ will denote the subspace of $C[0,1]$ consisting of those functions which vanish at 0, and $BV[0, 1]$ will denote the set of all functions of bounded variation on [0, 1].

LEMMA 1. *If* $\alpha(x)\in BV[0, 1]$ *and* $f\in C_0[0, 1]$, *then the function*

$$h(t) = \int_0^t \alpha(x)\,df(x)$$

is in $C_0[0, 1]$.

Proof. We may assume that $\alpha(x)$ is increasing on [0, 1]. By the formula for integration by parts we have

$$h(t) = \alpha(t)f(t)-\int_0^t f(x)\,d\alpha(x)$$

(since $f(0) = 0$). Fix any $t_0\in[0, 1)$. Then

$$\lim_{t\to t_0^+} h(t) = \lim_{t\to t_0^+} [\alpha(t)f(t) - \int_0^{t_0} f(x)\,d\alpha(x) - \int_{t_0}^{t} f(x)\,d\alpha(x)]$$

$$= \alpha(t_0^+)f(t_0) - \int_0^{t_0} f(x)\,d\alpha(x) - \lim_{t\to t_0^+} \int_{t_0}^{t} f(x)\,d\alpha(x).$$

Since $\alpha\uparrow$ and f is continuous, for any $t > t_0$ there exists $t_1 \in (t_0, t)$ such that

$$\int_{t_0}^{t} f(x)\,d\alpha(x) = f(t_1)\big(\alpha(t) - \alpha(t_0)\big),$$

and where $t_1 \to t_0$ as $t \to t_0$. Therefore

$$\lim_{t\to t_0} \int_{t_0}^{t} f(x)\,d\alpha(x) = \lim_{t\to t_0} f(t_1)\big(\alpha(t) - \alpha(t_0)\big) = f(t_0)\big(\alpha(t_0^+) - \alpha(t_0)\big),$$

so

$$h(t_0^+) = \alpha(t_0^+)f(t_0) - \int_0^{t_0} f(x)\,d\alpha(x) - f(t_0)\,\alpha(t_0^+) + f(t_0)\,\alpha(t_0)$$

$$= f(t_0)\,\alpha(t_0) - \int_0^{t_0} f(x)\,d\alpha(x) = h(t_0)$$

(by the above). Similarly, $h(t_0^-) = h(t_0)$ for $t_0 \in (0, 1]$, $h(0^+) = h(0)$, and $h(1^-) = h(1)$. Therefore $h(t)$ is continuous on $[0, 1]$, and since $h(0) = 0$, the lemma is proved.

It follows from Lemma 1 that for any $\alpha(x) \in BV[0, 1]$ we can define a linear operator T on $C_0[0, 1]$ by the formula

$$Tf(t) = \int_0^t \alpha(x)\,df(x).$$

We note next that T is continuous.

LEMMA 2. *If $\alpha \in BV[0, 1]$, then T is bounded.*

Proof. Let $\alpha = \alpha_1 - \alpha_2$, where α_i is increasing on $[0, 1]$ for $i = 1, 2$. As in the proof of Lemma 1, for any t we have

$$|Tf(t)| = |\alpha(t)f(t) - \int_0^t f(x)\,d\alpha(x)|$$

$$\leqslant |\alpha(t)|\,|f(t)| + \Big|\int_0^t f(x)\,d\alpha_1(x)\Big| + \Big|\int_0^t f(x)\,d\alpha_2(x)\Big|$$

$$\leqslant |\alpha(t)|\,|f(t)| + \int_0^t |f(x)|\,d\alpha_1(x) + \int_0^t |f(x)|\,d\alpha_2(x)$$

$$\leqslant \|f\|\,\|\alpha\| + \|f\|\,\mathrm{Var}\,\alpha_1 + \|f\|\,\mathrm{Var}\,\alpha_2$$

(where $\|\cdot\|$ denotes the norm and Var denotes the total variation). Therefore

$$\|Tf\| \leqslant (\|\alpha\| + \operatorname{Var}\alpha_1 + \operatorname{Var}\alpha_2)\|f\|,$$

and T is bounded on $C_0[0, 1]$.

LEMMA 3. *If* $\alpha \in BV[0, 1]$, *then*

$$T\phi_n = \int_0^t \alpha(x)\, h_n(x)\, dx \quad \textit{for } n = 1, 2, 3, \ldots$$

Proof. For each $n = 1, 2, 3, \ldots$ let $I_n = [a_n, b_n]$ denote the interval on which ϕ_n is supported and set $c_n = (a_n + b_n)/2$. If $n = 2^k + l$ for $k \geqslant 0$ and $0 \leqslant l < 2^k$, then by definition ϕ_n is linear on $[a_n, c_n]$ with $\phi_n'(x) = 2^k$ for $x \in (a_n, c_n)$, and with one-sided derivatives at a_n and at c_n also equal to 2^k. Therefore, if $t \in [a_n, c_n]$ then

$$\int_0^t \alpha(x)\, d\phi_n(x) = \int_0^t \alpha(x)\, \phi_n'(x)\, dx = \int_{a_n}^t \alpha(x)\, 2^k dx = \int_0^t \alpha(x)\, h_n(x)\, dx$$

(since $h_n(x) = 2^k$ for $x \in (a_n, c_n)$ and changing the value of the integrand at a_n or c_n does not affect the value of the Riemann integral). Similarly, $\phi_n'(x) = -2^k$ for $x \in [c_n, b_n]$, so for $t \in [c_n, b_n]$ we have

$$\int_0^t \alpha(x)\, d\phi_n(x) = \int_{a_n}^{c_n} \alpha(x)\, h_n(x)\, dx + \int_{c_n}^t \alpha(x)\, h_n(x)\, dx = \int_0^t \alpha(x)\, h_n(x)\, dx.$$

In any case, then,

$$T\phi_n(t) = \int_0^t \alpha(x)\, h_n(x)\, dx \qquad \text{for } t \in [0, 1] \text{ and } n = 1, 2, 3, \ldots$$

We now show that T is invertible as long as α is increasing and bounded away from 0, and hence that the set $\{\int_0^t \alpha(x)\, h_n(x)\, dx\}_{n=1}^{\infty}$ is a basis for $C_0[0, 1]$ equivalent to the basis $\{\phi_n\}_{n=1}^{\infty}$.

THEOREM 1. *If* $\alpha(x)$ *is increasing and* $\alpha(x) \geqslant \varepsilon > 0$ *for* $x \in [0, 1]$, *then the operator* T *on* $C_0[0, 1]$ *defined by*

$$Tf(t) = \int_0^t \alpha(x)\, df(x)$$

is invertible.

Proof. By Lemma 2, T is continuous. As in the proof of Lemma 1,

$$Tf(t) = \alpha(t) f(t) - \int_0^t f(x)\, d\alpha(x) = \alpha(t)\left[f(t) - \frac{1}{\alpha(t)} \int_0^t f(x)\, d\alpha(x) \right],$$

where the function

$$\frac{1}{\alpha(t)}\int_0^t f(x)\,d\alpha(x)$$

is in the space $B[0, 1]$ of all bounded functions on $[0, 1]$ since $\alpha(x) \geqslant \varepsilon > 0$ for x in $[0, 1]$.

Let J: $C_0[0, 1] \to B[0, 1]$ be the injection map and Q: $C_0[0, 1] \to B[0, 1]$ the operator defined by

$$Qf(t) = \int_0^t f(x)\,d\alpha(x).$$

CLAIM 1. *Q is a compact operator.*

According to [2], p. 260, a bounded set K in $B[0, 1]$ is relatively compact if and only if for every $\varepsilon > 0$ there exist disjoint measurable sets $\{E_i\}_{i=1}^n$ in $[0, 1]$ for which

$$\bigcup_{i=1}^n E_i = [0, 1]$$

and points s_i in E_i for $i = 1, 2, \ldots, n$ such that

$$\sup_{s\in E_i} |f(s_i)-f(x)| < \varepsilon \quad \text{for all } f\in K \text{ and all } i = 1, 2, \ldots, n.$$

Since α is increasing on $[0, 1]$, given $\varepsilon > 0$ we can partition $[\alpha(0), \alpha(1)]$ into subintervals $\{[y_{i-1}, y_i]\}_{i=1}^m$ of length $< \varepsilon$ and define $I_i = \alpha^{-1}(y_{i-1}, y_i)$ and $J_i = \alpha^{-1}(y_i)$ for $i = 1, 2, \ldots, m$, where each of these is then an interval, a point, or the empty set. Then the collection

$$\{A_j\}_{j=1}^p = \Big(\bigcup_{i=1}^m I_i\Big) \cup \Big(\bigcup_{i=1}^m J_i\Big)$$

is a disjoint collection of measurable sets in $[0, 1]$ whose union is $[0, 1]$. Note that if $t_1 < t_2$, and both are in some A_j, then

$$\operatorname*{Var}_{[t_1,t_2]} \alpha = \alpha(t_2)-\alpha(t_1) < \varepsilon$$

by construction. For any $j = 1, 2, \ldots, p$, choose a point s_j in A_j. Then if $t\in A_j$ and $f\in C_0[0, 1]$, with $\|f\| \leqslant 1$, we have (assuming $s_j \leqslant t$)

$$\Big|\int_0^t f(x)\,d\alpha(x)-\int_0^{s_j} f(x)\,d\alpha(x)\Big| = \Big|\int_{s_j}^t f(x)\,d\alpha(x)\Big| \leqslant \|f\|\operatorname*{Var}_{[s_j,t]}\alpha < \varepsilon.$$

Hence by the criterion noted above the set $K = \{Qf\colon \|f\| \leqslant 1, f\in C_0[0, 1]\}$ is relatively compact in $B[0, 1]$, and so Q is a compact operator from $C_0[0, 1]$ into $B[0, 1]$.

It follows that if S: $C_0[0, 1] \to B[0, 1]$ is defined by

$$Sf(t) = \frac{1}{\alpha(t)}(Qf)(t),$$

then S is also compact.

CLAIM 2. *$J-S$ is one-to-one.*

For, if $(J-S)(f) = 0$ for some $f \in C_0[0, 1]$, then $Tf = \alpha[J-S](f) = 0$ also (as a function in $C_0[0, 1] \subset B[0, 1]$). That is, by the above,

$$\alpha(t) f(t) = \int_0^t f(x)\, d\alpha(x) \qquad \text{for all } t \in [0, 1].$$

Since α is increasing, by the First Mean Value Theorem for integrals for each $t \in [0, 1]$ there exists a point x_t in $[0, t]$ such that

$$\int_0^t f(x)\, d\alpha(x) = f(x_t)[\alpha(t) - \alpha(0)].$$

Therefore for each t we have

$$f(t) = \frac{f(x_t)}{\alpha(t)}[\alpha(t) - \alpha(0)] \leqslant f(x_t)\left(1 - \frac{\alpha(0)}{\alpha(1)}\right),$$

where $\alpha(0) = \varepsilon > 0$. In particular, then, for any t,

$$|f(t)| \leqslant \|f\|\left(1 - \frac{\alpha(0)}{\alpha(1)}\right),$$

from which it follows that $\|f\| \leqslant (1 - \alpha(0)/\alpha(1))\|f\|$, a contradiction since $1 - \alpha(0)/\alpha(1) < 1$. Hence $J-S$ must be one-to-one.

CLAIM 3. *$J-S$ is bounded below on $C_0[0, 1]$.*

For, suppose $\|f_n\| = 1$ for $n = 1, 2, \ldots$ but $\|Jf_n - Sf_n\| \to 0$ (in $B(0, 1]$). Since S is compact, we may assume $\{Sf_n\}_{n=1}^{\infty}$ converges to some function g_0 in $B[0, 1]$. That is,

$$\|Jf_n - Sf_n\| \to 0 \qquad \text{and} \qquad \|Sf_n - g_0\| \to 0,$$

so $\{Jf_n\}_{n=1}^{\infty}$ converges to g_0 in $B[0, 1]$. Now $\{Jf_n\}_{n=1}^{\infty} \subset C_0[0, 1]$, a closed subset of $B[0, 1]$, so it follows that $g_0 \in C_0[0, 1]$ as well and that $\{f_n\}$ then converges to g_0 in $C_0[0, 1]$. But then $\{Sf_n\}_{n=1}^{\infty} \to Sg_0$ in $B[0, 1]$, while we already have $\{Sf_n\}_{n=1}^{\infty} \to g_0$ in $B[0, 1]$, where $g_0 = Jg_0$. That is, $Sg_0 = Jg_0$ or $(J-S)g_0 = 0$. But $\|g_0\| = 1$ since $\|f_n\| = 1$ for all n, $\|Jf_n\| = \|f_n\|$, and $\{Jf_n\}_{n=1}^{\infty} \to g_0$. This contradicts the fact established above that $J-S$ is one-to-one, so it must be that $J-S$ is bounded below on $C_0[0, 1]$. Since

$$Tf(t) = \alpha(t)[Jf(t) - Sf(t)] \qquad \text{and} \qquad \alpha(t) \geqslant \varepsilon > 0 \qquad \text{for all } t,$$

it follows that T is also bounded below on $C_0[0, 1]$, and hence that ran T is some closed subspace of $C_0[0, 1]$.

CLAIM 4. ran $T = C_0[0, 1]$.

To see that this is the case first note that since $\{\phi_n\}_{n=1}^{\infty}$ is a basis for $C_0[0, 1]$, it follows (by the above) that $\{T\phi_n\}_{n=1}^{\infty}$ is a basis for the range of T. If we define $V: L^1[0, 1] \to C[0, 1]$ by

$$Vg(t) = \int_0^t \alpha(x) g(x)\, dx,$$

then

$$Vh_n = \int_0^t \alpha(x) h_n(x)\, dx = T\phi_n \quad \text{for all } n \geqslant 1.$$

Since $\{h_n\}_{n=1}^{\infty}$ is a basis for $L^1[0, 1]$, the set $\{Vh_n\}_{n=1}^{\infty}$ has a linear span dense in ran V, so it follows that the closure of the range of V is precisely ran T. In particular, $[\text{ran } V]^{\perp} = [\text{ran } T]^{\perp}$.

Now let $\mu \in [\text{ran } V]^{\perp} \subset C[0, 1]^*$, identified as the space of regular Borel measures on $[0, 1]$. Then

$$\int_0^1 Vg(t)\, d\mu(t) = 0 \quad \text{for all } g \in L^1[0, 1],$$

or

$$\int_0^1 \int_0^t \alpha(x) g(x)\, dx\, d\mu(t) = 0 \quad \text{for all } g \in L^1[0, 1].$$

If $\chi_{[0,t]}$ denotes the characteristic function of $[0, t]$, this last can be written as

$$\int_0^1 \int_0^1 \chi_{[0,t]}(x)\, \alpha(x) g(x)\, dx\, d\mu(t) = 0 \quad \text{for all } g \in L^1[0, 1].$$

But then a trivial application of Fubini's Theorem ([3], p. 155) allows an interchange of the order of these integrals and we have

$$\int_0^1 \alpha(x) g(x) \int_0^1 \chi_{[0,t]}(x)\, d\mu(t)\, dx = 0 \quad \text{for all } g \in L^1(0, 1].$$

Again, this simply says

$$\int_0^1 \alpha(x) g(x)\, \mu[x, 1]\, dx = 0 \quad \text{for all } g \in L^1[0, 1],$$

and it follows that the function $\alpha(x)\, \mu[x, 1]$ must be 0 almost everywhere (w.r.t. Lebesgue measure) on $[0, 1]$. Since $\alpha(x) \geqslant \varepsilon > 0$ for all $x \in [0, 1]$, it then follows that $\mu[x, 1] = 0$ a.e. in $[0, 1]$, and a standard argument using the countable additivity of the integral establishes that $\mu(x, 1] = 0$ for all $x \geqslant 0$, hence that $\mu(a, b] = 0$ for all $0 \leqslant a < b \leqslant 1$. If we let β denote the normalized function of bounded variation on $[0, 1]$ associated with the measure μ, then

$$\beta(a) = \mu(\{a\}), \qquad \beta(b) = \mu(0, b] = 0 \text{ for } 0 < b \leqslant 1,$$

and

$$\int_0^1 f(x)\, d\mu(x) = \int_0^1 f(x)\, d\beta(x) = [-\mu(\{a\})] f(0) \qquad \text{for all } f \text{ in } C[0, 1].$$

That is, μ is a multiple of the point mass δ_0 at the point $x = 0$, so $[\operatorname{ran} V]^{\perp} = [\delta_0]$, and by our previous discussion it follows that $[\operatorname{ran} T]^{\perp} = [\delta_0]$ also. In other words, $\operatorname{ran} T = C_0[0, 1]$, and our claim is established.

Since we have shown that T is bounded below on $C_0[0, 1]$ and that T maps onto $C_0[0, 1]$, T is invertible on $C_0[0, 1]$ by the open mapping theorem, and the theorem is proved. From this we get the result on integral transforms of Haar functions we mentioned earlier:

THEOREM 2. *Let $\alpha(x)$ be an increasing function on $[0, 1]$ and $\alpha(x) \geqslant \varepsilon > 0$ for all x in $[0, 1]$. Then the set of functions $\{1, \int_0^t \alpha(x) h_n(x)\, dx\}_{n=1}^{\infty}$ is a basis for $C[0, 1]$ which is equivalent to the basis $\{\phi_n\}_{n=0}^{\infty}$.*

Proof. By Theorem 1, T is invertible. Since

$$T\phi_n = \int_0^t \alpha(x) h_n(x)\, dx \qquad \text{for all } n \geqslant 1$$

(Lemma 3) and since $\{\phi_n\}_{n=1}^{\infty}$ is a basis for $C_0[0, 1]$, it follows that $\{\int_0^t \alpha(x) h_n(x)\, dx\}_{n=1}^{\infty}$ is a basis for $C_0[0, 1]$ which is equivalent to $\{\phi_n\}_{n=1}^{\infty}$. But then since $\phi_0(x) = 1$, it follows trivially that the set $\{1, \int_0^t \alpha(x) h_n(x)\, dx\}_{n=1}^{\infty}$ is a basis for $C[0, 1]$ which is equivalent to the basis $\{\phi_n\}_{n=0}^{\infty}$, and the proof is complete.

Using an analogous argument we can also show that the bases for $C[0, 1]$ considered by Radecki [4] (where $\alpha(x)$ is taken to be a positive, continuous function in $BV[0, 1]$) are also all equivalent to Schauder's basis $\{\phi_n\}_{n=0}^{\infty}$.

THEOREM 3. *Let $\alpha(x)$ be a positive, continuous function in $BV[0, 1]$. Then the set $\{1, \int_0^1 \alpha(x) h_n(x)\, dx\}_{n=1}^{\infty}$ is a basis for $C[0, 1]$ which is equivalent to $\{\phi_n\}_{n=0}^{\infty}$.*

Proof. The fact that $\{1, \int_0^t \alpha(x) h_n(x)\,dx\}_{n=1}^{\infty}$ is some basis for $C[0, 1]$ was proved by Radecki [4], as we remarked above. As in the proof of Theorem 1 we have

$$Tf(t) = \alpha(t)f(t) - \int_0^t f(x)\,d\alpha(x) \quad \text{for } f \in C_0[0, 1] \text{ and all } t \in [0, 1].$$

If we let $M_\alpha \colon C_0[0, 1] \to C_0[0, 1]$ denote the operator of multiplication on $C_0[0, 1]$ by the function $\alpha(x)$, then since α is continuous and bounded away from 0, it follows that M_α is invertible and, by the above, $T = M_\alpha - Q$, where $Q \colon C_0[0, 1] \to C_0[0, 1]$ is defined by

$$Qf(t) = \int_0^t f(x)\,d\alpha(x).$$

Now it is easy to see that Q is a compact operator on $C_0[0, 1]$. For, if $f \in C_0[0, 1]$, $\|f\| \leqslant 1$, and $t_1 < t_2$, then

$$|Qf(t_2) - Qf(t_1)| = \Big|\int_{t_1}^{t_2} f(x)\,d\alpha(x)\Big| \leqslant \|f\| \operatorname*{Var}_{[t_1,t_2]} \alpha = V(t_2) - V(t_1),$$

where V is the variation function associated with α. But V is continuous on $[0, 1]$ since α is, and hence also uniformly continuous there. Therefore, given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $t_1 < t_2$ and $|t_2 - t_1| < \delta$, then $V(t_2) - V(t_1) < \varepsilon$, from which it follows that $|Qf(t_2) - Qf(t_1)| < \varepsilon$ whenever $\|f\| \leqslant 1$ and $|t_2 - t_1| < \delta$, so the set $\{Qf\colon \|f\| \leqslant 1\}$ is an equicontinuous subset of $C[0, 1]$. Being bounded in $C[0, 1]$, this set is then relatively compact by the Ascoli–Arzelà Theorem ([2], p. 266), and the operator Q is therefore compact on $C_0[0, 1]$.

That is, $T = M_\alpha - Q$, where M_α is invertible and Q is compact on $C_0[0, 1]$. Since we observed earlier that T is one-to-one, by the Fredholm Alternative T is invertible on $C_0[0, 1]$ and it follows that the basis $\{\int_0^t \alpha(x) h_n(x)\,dx\}_{n=1}^{\infty}$ for $C[0, 1]$ is equivalent to $\{\phi_n\}_{n=1}^{\infty}$. But then, as in Theorem 2, it trivially follows that the basis $\{1, \int_0^t \alpha(x) h_n(x)\,dx\}_{n=1}^{\infty}$ for $C[0, 1]$ is equivalent to the basis $\{\phi_n\}_{n=0}^{\infty}$, and the proof is complete.

REFERENCES

[1] Z. Ciesielski, *On Haar functions and on the Schauder basis of the space* $C[0, 1]$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7 (1959), pp. 227–232.

[2] N. Dunford and J. Schwartz, *Linear Operators. I*, Interscience Publishers, New York 1963.

[3] M. Munroe, *Measure and Integration*, 2nd ed., Addison-Wesley, Reading, Mass., 1971.
[4] J. Radecki, *Schauder bases in the space of continuous functions*, Comment. Math. Prace Mat. 13 (1970), pp. 193–196.
[5] I. Singer, *Bases in Banach Spaces. I*, Springer-Verlag, New York 1970.

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