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ON FILLING AN IRREDUCIBLE CONTINUUM WITH THE CARTESIAN PRODUCT OF AN ARC WITH A SIMPLE TRIOD

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In this paper, a *continuum* is a compact connected metric space. As defined in [3], $\mathscr{K}$ denotes the class of all continua K such that there exists an upper semicontinuous decomposition G of an irreducible continuum M with each element of G homeomorphic to K and with decomposition space M/G an arc. In [3] it is shown that the simple closed curve is not in $\mathscr{K}$ and, indeed, no finite 1-polyhedron except an arc is in $\mathscr{K}$. In [2], it is shown that neither the annulus nor the torus is in $\mathscr{K}$. A reasonable question at this point is that if K is not in $\mathscr{K}$, does $[0, 1]\times K$ not belong to $\mathscr{K}$? In this paper it is shown that the Cartesian product of an arc with a simple triod is in $\mathscr{K}$.

THEOREM. *The Cartesian product of the interval* $[0, 1]$ *with a simple triod is in* $\mathscr{K}$.

Proof. Let C denote the Cantor middle third set on $[0, 1]$ and $I_1, I_2, I_3, \ldots$ denote the components of $[0, 1]-C$. For each positive integer i, let c_i denote the left endpoint of $\bar{I}_i$, and d_i denote the right endpoint of $\bar{I}_i$.

By Theorem 1 of [1], there exists a compact metric continuum M such that

(1) M is irreducible and a subset of $[0, 1]\times[0, 1]$;

(2) there exists an upper semicontinuous collection G of arcs filling up M such that M/G is an arc;

(3) there exists a countable subcollection H of G such that

(a) if h is in H, h contains an arc Z_h such that each point of Z_h is a separating point of M and Z_h contains every separating point of M in h,

(b) if E' denotes the set of all points P such that P is an endpoint of Z_h for some h in H, then E' is dense in $M-\bigcup_{h\in H} Z_h$,

(c) if $\delta > 0$, then only finitely many members h of H have $\operatorname{diam}(Z_h) < \delta$,

(d) $\bigcup_{h\in H} Z_h$ contains all separating points of M.

As in the proof of Theorem 2 of [1], let K denote the collection to which k belongs if and only if, for some element h of H, k is the closure of a component of $h-Z_h$. Now

$$(G-H)^* \cup K^* = \overline{M - \bigcup_{h\in H} Z_h}$$

and $(G-H)\cup K$ is an upper semicontinuous collection of mutually exclusive arcs filling up $\overline{M-\bigcup_{h\in H} Z_h}$. Furthermore, the set

$$(\overline{M-\bigcup_{h\in H} Z_h})/[(G-H)\cup K]$$

is a copy of the Cantor set T.

Let f denote a continuous mapping from M onto $[0, 1]$ such that f restricted to $\bigcup_{h\in H} Z_h$ is one-to-one and onto $\bigcup_{n=1}^{\infty} \bar{I}_i$ and if $h\in H$, there is one and only one i such that $f(Z_h)=\bar{I}_i$.

If P is a point of $\overline{\{c_1, c_2, \ldots\}}$, there is an arc h_p of H such that $f(h_p)$ contains P. The set $h_p - Z_{h_p}$ has two components, r_p and s_p, where $f(r_p)=c_n$ and $f(s_p)=d_n$ for some n. Let a_p and a'_p denote the endpoints of r_p, where a_p is an endpoint of h_p; and let b_p and b'_p denote the endpoints of s_p, where b_p is an endpoint of h_p.

Let T denote a simple triod and J denote the junction point of T. Let t_1, t_2 and t_3 denote three arcs such that

$$t_1 \cup t_2 \cup t_3 = T$$

and, for each i, $1\leqslant i\leqslant 3$, t_i is the closure of a component of $T-J$.

Let us put

$$L_1 = \overline{M-\bigcup_{h\in H} Z_h} \times T.$$

If P is a point of C, let $L_1(P)$ denote $f^{-1}(P)\times T$. For each i, let $L'_1(c_i)$ denote $a'_{c_i}\times T$ and $L'_1(d_i)$ denote $b'_{c_i}\times T$.

Let us put

$$R = [[0, 1/2]\times T] \cup [[1/2, 1]\times (t_1\cup t_2)].$$

Note that while R is not homeomorphic to $[0, 1]\times T$, there is a retraction r of $[0, 1]\times T$ onto R; that is, r is a continuous map from $[0, 1]\times T$ onto R such that if x is a point of R, then $r(x)=x$. We can assume that $\operatorname{diam}(r^{-1}(x))<1$ for $x\in R$.

For each i, let α_{c_i} denote a proper subarc of $f^{-1}(c_i)$ in $\overline{M - \bigcup_{h \in H} Z_h}$ such that

(1) α_{c_i} contains a_{c_i};

(2) $\operatorname{diam}(f^{-1}(c_i) - \alpha_{c_i}) < 1/i$.

Also, for each i, let α_{d_i} denote a proper subarc of $f^{-1}(d_i)$ in $\overline{M - \bigcup_{h \in H} Z_h}$ such that

(1) α_{d_i} contains b_{c_i};

(2) $\operatorname{diam}(f^{-1}(d_i) - \alpha_{d_i}) < 1/i$.

Let $\theta_1, \theta_2, \theta_3, \ldots$ denote a sequence of arcs lying in T such that

(1) for each i, θ_i is $t_1 \cup t_2$, $t_2 \cup t_3$, or $t_1 \cup t_3$;

(2) $\bigcup_{i=1}^{\infty} (a'_{c_i} \times \theta_i)$ and $\bigcup_{i=1}^{\infty} (b'_{c_i} \times \theta_i)$ are dense in L_1.

For each i, put

$$R_{c_i} = [\alpha_{c_i} \times T] \cup [(f^{-1}(c_i) - \alpha_{c_i}) \times \theta_i]$$

and

$$R_{d_i} = [\alpha_{d_i} \times T] \cup [(f^{-1}(d_i) - \alpha_{d_i}) \times \theta_i].$$

Also, for each i, let r_{c_i} denote a retraction of $f^{-1}(c_1) \times T$ onto R_{c_i} such that if x is in R_{c_i}, then

$$\operatorname{diam}(r_{c_i}^{-1}(x)) < 2/i;$$

and let r_{d_i} denote a retraction of $f^{-1}(d_i) \times T$ onto R_{d_i} such that if x is in R_{d_i}, then

$$\operatorname{diam}(r_{d_i}^{-1}(x)) < 2/i.$$

Let U_1 denote the collection to which x belongs if and only if:

(1) for some point P of $C - \bigcup_{i=1}^{\infty} (c_i \cup d_i)$, x is a point of $f^{-1}(P) \times T$;

(2) for some i and some point P of R_{c_i}, x is $r_{c_i}^{-1}(P)$; or

(3) for some i and some point P of R_{d_i}, x is $r_{d_i}^{-1}(P)$.

U_1 is an upper semicontinuous collection of mutually exclusive closed point sets filling up L_1, since if, for each i, P_i is a point of R_{c_i} and Q_i is a point of R_{d_i}, then

$$\lim_{i \to \infty} \operatorname{diam}(r_{c_1}^{-1}(P_i)) = 0.$$

Let L_2 denote L_1/U_1.

If P is a point of $C - \bigcup_{i=1}^{\infty} (c_i \cup d_i)$, then $L_1(P)/U_1$ is homeomorphic to

$[0, 1] \times T$. If P is a point of $\bigcup_{i=1}^{\infty} (c_i \cup d_i)$, then $L_1(P)/U_1$ is homeomorphic to R.

Let $j_1, j_2, \ldots$ denote a sequence of points such that:

(1) for each i, j_i is in θ_i minus its endpoints;

(2) $\bigcup_{i=1}^{\infty} (a'_{c_i} \times j_i)$ and $\bigcup_{i=1}^{\infty} (b'_{c_i} \times j_i)$ are dense in L_1.

Also, let $\gamma_{c_1}, \gamma_{c_2}, \ldots$ denote a sequence of arcs such that:

(1) for each i, γ_{c_i} is a subset of $L(c_i)/U_1$;

(2) one endpoint of γ_{c_i} is $(a'_{c_i} \times j_i)/U_1$;

(3) the other endpoint is the endpoint of $(\alpha_{c_i} \times J)/U_1$ that is distinct from $(a_{c_i} \times J)/U_1$;

(4) γ_{c_i} does not intersect the union of all edges of $L(c_i)/U_1$ except at $(a'_{c_i} \times j_i)/U_1$;

(5) γ_{c_i} does not intersect $(\alpha_{c_i} \times J)/U_1$ except at one endpoint.

Similarly, define a sequence of arcs $\gamma_{d_1}, \gamma_{d_2}, \ldots$ such that for each i:

(1) γ_{d_i} is a subset of $L(d_i)/U_1$;

(2) one endpoint of γ_{d_i} is $(b'_{c_i} \times j_i)/U_1$;

(3) the other endpoint is the endpoint of $(\alpha_{d_i} \times J)/U_1$ that is distinct from $(b_{c_i} \times J)/U_1$;

(4) γ_{d_i} does not intersect the union of all edges of $L(d_i)/U_1$ except at $(b'_{c_i} \times j_i)/U_1$;

(5) γ_{d_i} does not intersect $(a_{d_i} \times J)/U_1$ except at one endpoint.

For each i, there exists an upper semicontinuous collection G_{c_i} of mutually exclusive closed point sets filling $L(c_i)/U_1$ such that:

(1) if g is in G_{c_i}, g is a point of $L(c_i)/U_1$ or a pair of points of $L(c_i)/U_1$;

(2) each point of $[(\alpha_{c_i} \times J)/U_1] \cup \gamma_{c_i}$ is an element of G_{c_i};

(3) if g is in G_{c_i}, $\operatorname{diam}(g) < 1/i$;

(4) $L(c_i)/U_1/G_{c_i}$ is homeomorphic to $[0, 1] \times T$;

(5) $(a'_{c_i} \times \theta_i)/U_1/G_{c_i}$ is a simple triod on the union of edges of $L(c_i)/U_1/G_{c_i}$ whose junction point g_{c_i} is within $1/i$ of $(a'_{c_i} \times j_i)/U_1/G_{c_i}$.

These collections can be constructed by sewing together pairs of points in the set

$$(f^{-1}(c_i) \times \theta_i)/U_1 - \{[(\alpha_{c_i} \times J)/U_1] \cup \gamma_{c_i}\}$$

along either side of the set $[(\alpha_{c_i} \times J)/U_1] \cup \gamma_{c_i}$ that are within $1/2i$ of $[(\alpha_{c_i} \times J)/U_1] \cup \gamma_{c_i}$.

Similarly, for each i there exists an upper semicontinuous collection G_{d_i} of mutually exclusive closed point sets filling R_{d_i}/U_1 such that

(1) if g is in G_{d_i}, then g is a point of $L(d_i)/U_1$ or a pair of points of $L(d_i)/U_1$;

(2) each point of $[(\alpha_{d_i} \times J)/U_1] \cup \gamma_{d_i}$ is an element of G_{d_i};

(3) if g is in G_{d_i}, then $d(g) < 1/i$;

(4) $L(d_i)/U_1/G_{d_i}$ is homeomorphic to $[0, 1] \times T$;

(5) $(b'_{c_i} \times \theta_i)/U_1/G_{d_i}$ is a simple triod on the union of edges of $L(d_i)/U_1/G_{d_i}$ whose junction point g_{d_i} is within $1/i$ of $(b'_{c_i} \times j_i)/U_1/G_{d_i}$.

Let U_2 denote the collection to which x belongs if and only if

(1) for some point P of $C - \bigcup_{i=1}^{\infty} (c_i \cup d_i)$, x is a point of $f^{-1}(P) \times T$;

(2) for some i, x is an element of G_{c_i}; or

(3) for some i, x is an element of G_{d_i}.

U_2 is an upper semicontinuous collection of mutually exclusive closed point sets filling L_1/U_1 such that if P is in C, then $L_1(P)/U_1/U_2$ is homeomorphic to $[0, 1] \times T$. Let L_3 denote $L_1/U_1/U_2$.

For each i, let T_{c_i} denote a simple triod lying in $(a'_{c_i} \times \theta_i)/U_1/U_2$ such that $\operatorname{diam}(T_{c_i}) < 1/i$ and T_{d_i} denote a simple triod lying in $(b'_{c_i} \times \theta_i)/U_1/U_2$ such that $\operatorname{diam}(T_{d_i}) < 1/i$. Note that $\bigcup_{i=1}^{\infty} g_{c_i}$, the union of all the junction points of $\bigcup_{i=1}^{\infty} T_{c_i}$, is dense in $L_1/U_1/U_2$ and $\bigcup_{i=1}^{\infty} g_{d_i}$ is dense in $L_1/U_1/U_2$. Also,

$$\lim_{i \to \infty} \operatorname{diam}(T_{c_i} \cup T_{d_i}) = 0.$$

For each i, let m_i denote a homeomorphism from T_{c_i} onto T_{d_i}. Let U_3 denote the collection to which x belongs if and only if:

(1) for some point P of $(L_1/U_1/U_2) - \bigcup_{i=1}^{\infty} (T_{c_i} \cup T_{d_i})$, x is P;

(2) for some positive integer i and some point P of T_{c_i}, x is $P \cup m_i(P)$.

Clearly, U_3 is an upper semicontinuous decomposition of $L_1/U_1/U_2$. Let L_4 denote $L_1/U_1/U_2/U_3$.

Finally, let U_4 denote the collection to which x belongs if and only if:

(1) for some point P of $C - \bigcup_{i=1}^{\infty} (c_i \cup d_i)$, x is $L_1(P)/U_1/U_2/U_3$; or

(2) for some positive integer i, x is $[L_1(c_i) \cup L_1(d_i)]/U_1/U_2/U_3$.

Each element of U_4 is homeomorphic to $[0, 1] \times J$ and L_4/U_4 is an arc. To see that L_4 is irreducible, all that is necessary is to observe that if N is a neighborhood of a point of L_4, then, for some i, N contains $T_{c_i} \cup m_i(T_{c_i})$, which is a simple triod in L_4 that separates

$$(f^{-1}(0)\times T)/U_1/U_2/U_3$$

from

$$(f^{-1}(1)\times T)/U_1/U_2/U_3.$$

Therefore, $[0, 1]\times T$ is in $\mathscr{K}$.

Comments. In a similar fashion it can be shown that if T is a finite 1-polyhedron which contains no simple closed curve, then $[0, 1]\times T$ is in $\mathscr{K}$. However, does $T\times T$ belong to $\mathscr{K}$? **(P 1342)**

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