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SUBSETS OF COMPACTA

BY

MICHAEL LAIDACKER (BEAUMONT, TEXAS)

In this paper* several results concerning subcompacta of continua are proved. In particular, it is proved that there is a compactum X such that every continuum is homeomorphic to a component of X. All compacta in this paper are metric.

THEOREM. *For every compactum X with a given metric, there is a compactum Y in $C\times X$ such that every subcompactum of X is isometric to $(\{c\}\times X)\cap Y$ for some $c\in C$, and $(\{c\}\times X)\cap Y$ is isometric to a subcompactum of X for every c in the Cantor set C.*

Proof. Let X be a compactum with a given metric d_X and let 2^X be the set of all subcompacta of X such that each element of 2^X is isometric to a subcompactum of X. Let the topology on 2^X be the exponential topology described in [2], p. 160. Let C denote the usual Cantor set with the usual metric d_C. Since X is a compact metric space, 2^X is a metric space. 2^X is compact, since X is compact. By [1], p. 127, there is an onto mapping $f\colon C\to 2^X$. Write

$$Y' = \big\{\{c\}\times Z\colon\ c\in C, Z\in 2^X, \text{ and } f(c)=Z\big\} \quad \text{and} \quad Y=\bigcup Y'.$$

We need only to show that Y is compact to complete the theorem. Define the distance in Y to be

$$d\big((c_1,x_1),(c_2,x_2)\big) = d_C(c_1,c_2)+d_X(x_1,x_2)$$

for (c_1,x_1) and (c_2,x_2) belonging to Y. Let $(c_1,x_1),(c_2,x_2),\ldots$ be an infinite sequence of points in Y. Since $C\times X$ is compact and $(c_i,x_i)\in C\times X$, there are a subsequence $(c_{i_1},x_{i_1}),(c_{i_2},x_{i_2}),\ldots$ and a point $(c,x)\in C\times X$ such that $(c_{i_n},x_{i_n})\to(c,x)$ as $n\to\infty$. Consequently, $c_{i_n}\to c$ as $n\to\infty$. Thus $f(c_{i_n})\to f(c)$ as $n\to\infty$. This implies that $\{c\}\times Z\subset Y$, where $f(c)=Z$. In fact, letting $f(c_{i_n})=Z_{i_n}$, we have

$$\{c_{i_n}\}\times Z_{i_n}\to\{c\}\times Z, \qquad \text{where } \{c_{i_n}\}\times Z_{i_n}\subset Y \text{ for } n\geqslant 1.$$

* The results in this paper are those contained in part of the fourth chapter of the author's doctoral dissertation, submitted to the University of Houston, 1976.

But $(c_{i_n}, x_{i_n}) \in \{c_{i_n}\} \times Z_{i_n}$. We infer, thus, that $(c_{i_n}, x_{i_n}) \to (c, x)$, where $(c, x) \in Y$. Therefore, Y is sequentially compact; and since Y is metric, Y is compact.

COROLLARY 1. *There is a compactum X such that every continuum is homeomorphic to a component of X.*

For the proof, let X be the Hilbert cube in the Theorem.

COROLLARY 2. *For every compactum X with a given metric, there is a compactum Y in $C \times X$ such that every subcontinuum of X is isometric to $(\{c\} \times X) \cap Y$ for some $c \in C$; and for every c belonging to the Cantor set, $(\{c\} \times X) \cap Y$ is either empty or isometric to a subcontinuum of X.*

Proof. By the Theorem, there exists a compactum Z in $C \times X$ such that every subcompactum of X is isometric to $(\{c\} \times X) \cap Z$ for some $c \in C$, and $(\{c\} \times X) \cap Z$ is isometric to a subcompactum of X for every c in the Cantor set C. Thus Z contains as components every subcontinuum of X such that each of these subcontinua is isometric to a subcontinuum of X. There exists a function $f\colon Z \to C$ such that, K and K' being components of Z, $K \neq K'$ implies $f(K) \neq f(K')$. Write $h(x) = \big(f(x), x\big)$ for every $x \in Z$. Then letting $Y = h(Z)$, we have the corollary.

REFERENCES

[1] J. G. Hocking and G. S. Young, *Topology*, Reading 1961.
[2] K. Kuratowski, *Topology I*, New York 1966.

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