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# A NON-COMMUTATIVE ORLICZ-PETTIS TYPE THEOREM

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In this note we prove an Orlicz-Pettis type theorem in non-commutative topological groups, thus extending partially the results of [1], [7], [4], and [8]. We proceed by reducing our problem to the situation which makes possible an application of an earlier result due to Kalton (see [3], Added in proof).

**1. Copies of the Cantor set $C = \{0, 1\}^N$ in uniform spaces.** Let $E$ be a set and $V$ a subset of $E \times E$. A subset $A$ of $E$ is said to be *$V$-bounded* if $A \times A \subset V$, and *countably $V$-bounded* if there is a sequence $(A_n)$ of $V$-bounded subsets of $E$ which covers $A$.

**1.1.** PROPOSITION. *Let $E$ be a hereditarily Lindelöf space and $V$ a subset of $E \times E$ such that $E$ is not countably $V$-bounded. Then there exists a closed subset $F$ of $E$ such that $E \setminus F$ is countably $V$-bounded and no nonempty open subset of $F$ is countably $V$-bounded.*

Proof. Let $B$ be the union of the family $\mathscr{B}$ of all countably $V$-bounded open subsets of $E$. Since $E$ is hereditarily Lindelöf, there is a sequence $(B_n)$ in $\mathscr{B}$ such that

$$B = \bigcup_{n=1}^{\infty} B_n.$$

It follows that $B \in \mathscr{B}$, and so $B$ is the largest countably $V$-bounded open subset of $E$. Then $F = E \setminus B$ is as required.

**1.2.** THEOREM. *Let $E$ be a Polish space and $V$ a closed subset of $E \times E$ containing the diagonal of $E \times E$ and such that $E$ is not countably $V$-bounded. Then $E$ contains a homeomorphic copy $K$ of the Cantor set which is $V$-discrete, i.e. $(x, y) \notin V$ for all $x, y \in K$ with $x \neq y$.*

Proof. In view of 1.1, we may assume that no nonempty open subset of $E$ is countably $V$-bounded. Let $d$ be a complete metric defining the topology of $E$. We define inductively, for each $n \in N$ and every sequence $e_1, \ldots, e_n$ with terms 0 or 1, a closed ball $K(e_1, \ldots, e_n)$ in $E$ so that the following conditions are satisfied:

(1) $K(e_1, \ldots, e_n, e_{n+1}) \subset K(e_1, \ldots, e_n)$;

(2) if $x \in K(e_1, \ldots, e_{n-1}, 0)$, $y \in K(e_1, \ldots, e_{n-1}, 1)$, then $(x, y) \notin V$ (in particular, these two balls are disjoint);

(3) the radius of the ball $K(e_1, \ldots, e_n)$ is less than or equal to $2^{-n}$.

We begin with $K(\emptyset) = E$ for $n = 0$. Suppose the balls $K(e_1, \ldots, e_i)$ have already been defined for all possible zero-one sequences $e_1, \ldots, e_i$, where $i \leqslant n$.

Let $(e_1, \ldots, e_n) \in \{0, 1\}^n$. Since the interior $K(e_1, \ldots, e_n)^\circ$ is not $V$-bounded, there are $a, b \in K(e_1, \ldots, e_n)^\circ$ such that $(a, b) \notin V$. Since $V$ is closed, there is an $r$, $0 < r \leqslant 2^{-n-1}$, such that the closed balls $B(a; r)$ and $B(b; r)$ are contained in $K(e_1, \ldots, e_n)$, and $x \in B(a; r)$, $y \in B(b; r)$ imply $(x, y) \notin V$. We put $K(e_1, \ldots, e_n, 0) = B(a; r)$ and $K(e_1, \ldots, e_n, 1) = B(b; r)$.

It is now clear that

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(e_j) \in \{0,1\}^n} K(e_1, \ldots, e_n) = \bigcup_{(e_j) \in \{0,1\}^N} \bigcap_{n=1}^{\infty} K(e_1, \ldots, e_n)$$

has the required properties.

Let $(X, \mathscr{U})$ be a uniform space. It is said to be *$\sigma$-bounded* if, for each vicinity $U \in \mathscr{U}$, $X$ is countably $U$-bounded. If $\varrho$ is a topology on $X$, then the uniformity $\mathscr{U}$ is said to be *$\varrho$-polar* if $\mathscr{U}$ has a base consisting of sets which are closed in $(X \times X, \varrho \times \varrho)$.

**1.3.** COROLLARY. *Let $f$ be a continuous map of a Polish space $E$ onto a Hausdorff space $(X, \varrho)$ and let $\mathscr{U}$ be a $\varrho$-polar uniformity on $X$ such that $X$ is not $\sigma$-bounded (with respect to $\mathscr{U}$). Then there are $U \in \mathscr{U}$ and a homeomorphic copy $K$ of the Cantor set in $E$ such that $(f(x), f(y)) \notin U$ for all $x, y \in K$ with $x \neq y$. (Hence $f|K$ is one-one, and so a homeomorphic embedding of $K$ into $(X, \varrho)$.)*

Proof. By assumption, there is a $\varrho$-closed vicinity $U \in \mathscr{U}$ such that $X$ is not countably $U$-bounded. Since the map $f \times f\colon E \times E \to X \times X$ which sends $(x, y) \in E \times E$ to $(f(x), f(y)) \in X \times X$ is continuous, the set $V = (f \times f)^{-1}(U)$ is a closed subset of $E \times E$ containing the diagonal of $E \times E$. Clearly, $E$ is not countably $V$-bounded. It remains to apply Theorem 1.2.

In particular, by taking any appropriate uniformity which induces the discrete topology in $E$, Corollary 1.3 yields the well-known result that a continuous map of a Polish space onto an uncountable Hausdorff space is homeomorphic on a copy of the Cantor set ([6], § 36 V). The reader will remark that our proof of Theorem 1.2 is a modification of the argument used in [6], loc. cit.

Finally, we note the following fact:

**1.4.** *Let two Hausdorff group topologies $\beta$ and $a$ be given on a group $G$. Then the following conditions are equivalent:*

(i) *One of the three (i.e. right, left or two-sided) uniformities of $(G, \beta)$ is $a$-polar.*

(ii) *Each of the three uniformities of $(G, \beta)$ is $a$-polar.*

(iii) *$\beta$ has a base of $a$-closed neighbourhoods of the unit.*

If one of these conditions is satisfied, we say that $\beta$ is *$a$-polar.*

**2. An Orlicz-Pettis type theorem.** Let $G = (G, a)$ be a Hausdorff topological group. A sequence $(x_n)$ in $G$ is said to be *subproduct convergent* if the product $\prod_{n=1}^{\infty} x_{k_n}$ converges for every increasing sequence $(k_n)$ in $N$. If this is so, we can define a continuous map $m: C \to G$ by the formula

$$m(1_A) = \prod_n x_{k_n},$$

where $1_A$ is the characteristic function of a set $A \subset N$ and $k_1, k_2, \ldots$ is the arrangement of the elements in $A$ in increasing order; $m(\emptyset) = e$, the unit of $G$. For another Hausdorff group topology $\beta$ on $G$, the map $m: C \to (G, \beta)$ is continuous if and only if the sequence $(x_n)$ is subproduct convergent in $(G, \beta)$ (cf. [3]).

Let us quote one of the possible conditions defining $B_r$-completeness of a Hausdorff topological group $(G, \beta)$: if a Hausdorff group topology $\varrho \subset \beta$ on $G$ is such that

$$\{\text{closure}_\varrho U\colon\ U \text{ is a } \beta\text{-neighbourhood of } e\}$$

is a base of $\varrho$-neighbourhoods of $e$, then $\varrho = \beta$ ([5], Theorem 31.4). It follows from a result of Brown ([2], Theorem 4) that every Čech complete group is $B_r$-complete. Recall that a topological space is said to be *Čech complete* if it is homeomorphic to a dense $G_\delta$-subset of a compact Hausdorff space. In particular, every complete metric group is $B_r$-complete.

Let $E$ be a Polish space. A subset $A \subset E$ is called a *Sierpiński-Marczewski set* if it has the following property:

(s) Any perfect subset of $E$ contains a perfect subset $P$ such that $P \subset A$ or $P \cap A = \emptyset$.

A function $f$ on $E$ into a topological space $F$ is *Sierpiński measurable* if, for any open set $B \subset F$, $f^{-1}(B)$ is a Sierpiński-Marczewski set in $E$. For a discussion of the Sierpiński measurability see [10].

**2.1.** Theorem. *Let $G$ be a Hausdorff topological group under topologies $\beta$ and $a$. Assume that $a \subset \beta$ and that one of the following conditions is satisfied:*

(A) *$\beta$ is $a$-polar,*

(B) *$(G, \beta)$ is $B_r$-complete.*

*Let $(x_n)$ be subproduct convergent in $(G, a)$ and let $m\colon C \to (G, a)$ be the corresponding continuous map. If the inclusion map $j\colon \big(m(C), a\big) \to (G, \beta)$ is Sierpiński measurable, then $j \circ m$ is continuous, i.e. $(x_n)$ is subproduct convergent in $(G, \beta)$.*

Proof. Observe that $m(C)$ is $\sigma$-bounded with respect to the right uniformity of $(G, \beta)$ restricted to $m(C) \times m(C)$. Otherwise, applying 1.2 with $E = \big(m(C), a\big)$, we could find a homeomorphic copy, $K$ say, of $C$ in $\big(m(C), a\big)$, which is discrete in $\big(m(C), \beta\big)$. Any subset of $(K, \beta)$ being open, we would conclude that any subset of $K$ is a Sierpiński-Marczewski set, which is false ([10], 2.2). Now, case (A) follows from Theorem 1 of [3] (cf. Remark and Added in proof). Case (B) can be proved in the same way as Theorem 2 (a) of [3], appealing only to the fact that $(G, \beta)$ is $B_r$-complete instead of Weston's result.

Remarks. 1. In particular, Theorem 2.1 applies under the following more usual measurability conditions:

(UM) The identity $i\colon (G, a) \to (G, \beta)$ is universally measurable [7].

(BP) The identity $i\colon (G, a) \to (G, \beta)$ has the Baire property in the restricted sense ([6], § 32, IV; in other words, $i$ is "universally" BP-measurable).

Indeed, universally measurable sets as well as those having the Baire property in the restricted sense are Sierpiński-Marczewski sets ([10], 5.2 and 5.1).

2. Let us note that though the Baire property part extends only partially the result of Andersen and Christensen [1], it does not follow from their result even for commutative groups. First of all it is not clear whether $j \circ m$ has the Baire property in the restricted sense if $j$ does so. Even if it is the case (as assumed in [1]), the reasoning of Andersen and Christensen cannot be applied as long as we do not know *a priori* that $m(C)$ is $\sigma$-bounded in $(G, \beta)$.

On the other hand, the $\sigma$-boundedness condition used in [1] happens to be superfluous. Indeed, there is a quite recent result of Prikry [9] which implies the following:

A BP-measurable map $f$ on $C$ into a metric space is essentially separably valued (i.e., there is a set $Q$ of the first Baire category such that $f(C \setminus Q)$ is separable).

This makes possible the reduction to the $\sigma$-bounded case in the theorem of Andersen and Christensen exactly in the same way as in [7], where Sazonov's older result was used in the (UM) setting.

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