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FREE $\mathfrak{m}$-PRODUCTS OF LATTICES. I

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It has long been believed that "finitary" lattice theory has a natural extension to an "$\mathfrak{m}$-ary" lattice theory (where $\mathfrak{m}$ is an infinite regular cardinal): replace the requirement that all finite nonempty subsets have meets and joins with the one that all nonempty subsets of less than $\mathfrak{m}$ elements have meets and joins.

It is the purpose of this paper* to carry out this program for free products of lattices. The emphasis is twofold: laying the groundwork by verifying all the basic results (following the exposition of free products by the first-named author in his book [5] on general lattice theory) and pointing out divergencies from the finitary case, thereby discovering new areas of research that have no counterpart in the finitary case.

In the first part of the paper we find the $\mathfrak{m}$-ary versions of the basic results. We also introduce and investigate two new sublattice concepts – intact and closure sublattices – that play an important role in the $\mathfrak{m}$-ary case.

0. Introduction. Throughout this paper, $\mathfrak{m}$ is an infinite regular cardinal. A lattice L is $\mathfrak{m}$-*complete* (1) (or L is an $\mathfrak{m}$-*lattice*) if, for any nonempty $S \subseteq L$ with $|S| < \mathfrak{m}$, the join and meet of S exist in L. The concepts of $\mathfrak{m}$-*sublattice*, $\mathfrak{m}$-*generated*, and $\mathfrak{m}$-*homomorphism* are defined in the natural way. (There is no need to allow a singular cardinal $\mathfrak{n}$ instead of $\mathfrak{m}$ in these definitions since any $\mathfrak{n}$-lattice is an $\mathfrak{n}^+$-lattice and any $\mathfrak{n}$-homomorphism is an $\mathfrak{n}^+$-homomorphism; cf. [2] and [13].) For example, the $\mathfrak{m}$-lattice L is $\mathfrak{m}$-*generated* by a subset X if L is the smallest $\mathfrak{m}$-sublattice of L that contains X. Whenever $\mathfrak{m}$ is omitted, it is understood that $\mathfrak{m} = \aleph_0$.

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(1) Our usage of the term "$\mathfrak{m}$-complete" is different from the customary usage in the literature.

Let $CF_{\mathfrak{m}}(X)$ denote the completely free $\mathfrak{m}$-lattice $\mathfrak{m}$-generated by a poset X. Crawley and Dean [2] found a structure theorem for $CF_{\mathfrak{m}}(X)$ and showed that if $|X| \leqslant \mathfrak{m}$, then $CF_{\mathfrak{m}}(X)$ is isomorphic to an $\mathfrak{m}$-sublattice of the free $\mathfrak{m}$-lattice on 3 generators. Jónsson [13] proved a Normal Form Theorem for the elements of $CF_{\mathfrak{m}}(X)$ expressed as polynomials in X. This latter result has been generalized in [8].

The *free* $\mathfrak{m}$*-product* of a family $(L_i \mid i \in I)$ of $\mathfrak{m}$-lattices is the (unique up to isomorphism) $\mathfrak{m}$-lattice L generated by $\dot{\bigcup}(L_i \mid i \in I)$ such that, for any $\mathfrak{m}$-lattice K and $\mathfrak{m}$-homomorphism $\varphi_i\colon L_i \to K$ $(i \in I)$, there is an $\mathfrak{m}$-homomorphism $\varphi\colon L \to K$ extending each φ_i $(i \in I)$. We shall show that most of the results in Chapter VI of Grätzer's book [5] concerning free products of lattices (the $\mathfrak{m} = \aleph_0$ case) can be generalized to free $\mathfrak{m}$-products of $\mathfrak{m}$-lattices. Our presentation will generally follow the order of [5]; we shall emphasize the differences with the finitary $(\mathfrak{m} = \aleph_0)$ case. (We direct the reader to [5] for further references to the finitary case.)

Since any free $\mathfrak{m}$-product of finite chains is also the completely free $\mathfrak{m}$-lattice $\mathfrak{m}$-generated by the disjoint union of these chains, we also present some results on completely free $\mathfrak{m}$-lattices that complement and contrast our results on free $\mathfrak{m}$-products. In particular, $F_{\mathfrak{m}}(\mathfrak{n})$, the free $\mathfrak{m}$-lattice on $\mathfrak{n}$ generators, is the completely free $\mathfrak{m}$-lattice generated by an antichain with $\mathfrak{n}$ elements.

Let $\mathfrak{m}' \leqslant \mathfrak{m}$ be an infinite regular cardinal and let $\mathscr{L} = (L_i \mid i \in I)$ be a family of $\mathfrak{m}$-lattices. Clearly, $\mathscr{L}$ is also a family of $\mathfrak{m}'$-lattices. If K is the free $\mathfrak{m}'$-product of $\mathscr{L}$ and L is the free $\mathfrak{m}$-product of $\mathscr{L}$, the natural $\mathfrak{m}'$-homomorphism $\varphi\colon K \to L$ (that maps each L_i identically) is one-to-one. Hence, K can be considered to be a subposet of L. However, φ does not necessarily preserve all existing $\mathfrak{m}$-joins in K. For example, if $\mathfrak{m}' = \aleph_0$, $\mathfrak{m} = \aleph_1$, and $\mathscr{L} = (L_0, L_1)$ with $L_0 = \omega + 1$ and $L_1 = \{e\}$, then

$$\bigvee (n \wedge e \mid n < \omega) = \omega \wedge e$$

in K, whereas the two sides of this equality represent distinct elements in L. (The Structure Theorem of Section 1 can be used to verify these facts.) This topic is studied in Section 2.

1. Free $\mathfrak{m}$-products of $\mathfrak{m}$-lattices. Considered as an algebra with infinitary operations, an $\mathfrak{m}$-lattice is a nonempty set endowed with the $\mathfrak{n}$-ary operations of join and meet for each $\mathfrak{n}$ $(0 < \mathfrak{n} < \mathfrak{m})$. However, following [2], we modify the usual definition of an infinitary polynomial and use only two infinitary operation symbols.

For an arbitrary set X, the set $P_{\mathfrak{m}}(X)$ of $\mathfrak{m}$*-polynomials in* X is(2) $X_{\mathfrak{m}}$, where the X_α $(\alpha \leqslant \mathfrak{m})$ are inductively defined as follows: $X_0 = X$; if $\alpha > 0$

(2) Note that we identify cardinals with initial ordinals.

and $Y = \bigcup(X_\beta \mid \beta < \alpha)$, then X_α consists of all elements of Y together with all expressions of the form $\bigvee S$ or $\bigwedge S$ (considered *formally*), where $S \subseteq Y$ and $0 < |S| < \mathfrak{m}$. Thus, $\bigvee S$ and $\bigvee T$ are equal exactly when $S = T$, and $\bigvee S$ and $\bigwedge T$ are never equal. For example, if $x \in X$, then x, $\bigvee\{x\}$, and $\bigwedge\{x\}$ are three distinct polynomials. The *rank* $\varrho(p)$ of an $\mathfrak{m}$-polynomial p is the least ordinal α such that $p \in X_\alpha$. Note that $\varrho(p) < \mathfrak{m}$ for every $\mathfrak{m}$-polynomial p.

Throughout this section, L is the free $\mathfrak{m}$-product of the fixed family $(L_i \mid i \in I)$ of pairwise disjoint $\mathfrak{m}$-lattices and X is the set $\bigcup(L_i \mid i \in I)$. We assume that $X \subseteq L$. Note that x and y are incomparable in X if $x \in L_i$ and $y \in L_j$ with $i \neq j$. The letters p, q, and r will always denote $\mathfrak{m}$-polynomials in X. Let $\bar{p}$ denote the *value* of p in L; we also say that p *represents* $\bar{p}$ and that p is a *representation* of $\bar{p}$. Observe that $\bar{p}$ is an element of L and has a natural inductive definition. The *rank* $\varrho(a)$ of $a \in L$ is the least ordinal $\varrho(p)$, where p represents a. (The rank of an element of L can also be defined without reference to $\mathfrak{m}$-polynomials; see [8] or [13].)

For any poset Q, $Q^b = Q \dot\cup \{0, 1\}$ is the poset obtained by adjoining a new zero and one to Q. If A is an $\mathfrak{m}$-lattice, then A^b is an $\mathfrak{m}$-lattice in which 0 is $\mathfrak{m}$-meet-irreducible and 1 is $\mathfrak{m}$-join-irreducible. For $i \in I$, the *lower i-cover* of p, denoted by $p_{(i)}$, is the element of $(L_i)^b$ that is obtained by replacing in p every element from $X - L_i$ by 0 and evaluating the resulting expression in $(L_i)^b$. We regard $(L_i)^b$ as a subposet of L^b; hence $p_{(i)} \in (L_i)^b \subseteq L^b$. The *upper i-cover*, $p^{(i)}$, is defined dually. Let φ: $L \to (L_i)^b$ be the $\mathfrak{m}$-homomorphism defined by $x\varphi = x$ if $x \in L_i$ and by $y\varphi = 0$ if $y \in X - L_i$. By induction on the rank, it follows that $p_{(i)} = \bar{p}\varphi$. We can now state the Structure Theorem for Free $\mathfrak{m}$-Products:

THEOREM 1.1. *Let the* $\mathfrak{m}$*-lattice* L *be the free* $\mathfrak{m}$*-product of the family* $(L_i \mid i \in I)$ *of* $\mathfrak{m}$*-lattices and let* $X = \dot\bigcup(L_i \mid i \in I)$. *For* $p, q \in P_\mathfrak{m}(X)$, *the relation* $p \subseteq q$ *holds iff it is a consequence of the following rules*:

(C) *For some* $i \in I$, $p^{(i)} \leqslant q_{(i)}$ *in* L_i.

($\wedge$W) $p = \bigwedge S$ *and* $s \subseteq q$ *for some* $s \in S$.

($\vee$W) $p = \bigvee S$ *and* $s \subseteq q$ *for all* $s \in S$.

(W$\wedge$) $q = \bigwedge T$ *and* $p \subseteq t$ *for all* $t \in T$.

(W$\vee$) $q = \bigvee T$ *and* $p \subseteq t$ *for some* $t \in T$.

(*This is an inductive definition on* $\langle \varrho(p), \varrho(q) \rangle$ *in the lexicographic order.*) *The relation* $p \subseteq q$ *holds iff* $\bar{p} \leqslant \bar{q}$ *in* L.

The finitary case of Theorem 1.1 was given by Grätzer et al. [11] (see also Jónsson [13]). To prove the Structure Theorem, we shall use essentially the same sequence of lemmas that were used in [5] for the finitary case.

A lower or upper cover that is distinct from both 0 and 1 is called *proper*.

LEMMA 1.1. *For any* $p \in P_{\mathfrak{m}}(X)$, $p_{(i)} \leqslant p^{(j)}$ *in* X^b *for any* $i, j \in I$. *Consequently, if both* $p_{(i)}$ *and* $p^{(j)}$ *are proper, then* $i = j$.

Proof. Instead of modifying the syntactical proof in [5], we give a proof analogous to that of Jónsson [13]. Recall that $p_{(i)} = \bar{p}\varphi$, where $\varphi\colon L \to (L_i)^b$ is the $\mathfrak{m}$-homomorphism introduced before the statement of Theorem 1.1. Let $K = \{a \in L \mid a\varphi \leqslant a \text{ in } L^b\}$. Since K contains X and K is $\mathfrak{m}$-complete, we conclude that $K = L$. Hence $p_{(i)} \leqslant \bar{p}$ in L^b for any $p \in P_{\mathfrak{m}}(X)$. Thus $p_{(i)} \leqslant \bar{p} \leqslant p^{(j)}$ in L^b and, therefore, $p_{(i)} \leqslant p^{(j)}$ in X^b.

We next establish some properties of the relation $\subseteq$ defined in the statement of Theorem 1.1.

LEMMA 1.2. *Let* $p, q, r \in P_{\mathfrak{m}}(X)$ *and* $i \in I$. *Then*

(i) $p \subseteq p$;

(ii) $p \subseteq q$ *implies that* $p_{(i)} \leqslant q_{(i)}$ *and* $p^{(i)} \leqslant q^{(i)}$;

(iii) $p \subseteq q$ *and* $q \subseteq r$ *imply that* $p \subseteq r$.

Proof. (i) If $\varrho(p) = 0$, then $p \in L_i$ for a unique $i \in I$. Since $p = p_{(i)} = p^{(i)}$, the containment $p \subseteq p$ holds by (C). Let $p = \bigwedge S$. Since $s \subseteq s$ holds for all $s \in S$ by induction on the rank, it follows by $(\wedge \mathrm{W})$ that $\bigwedge S \subseteq s$ for all $s \in S$. Hence, applying $(\mathrm{W} \wedge)$, we get $p = \bigwedge S \subseteq \bigwedge S = p$.

(ii) If $p \subseteq q$ by (C), then $p^{(j)} \leqslant q_{(j)}$ for some $j \in I$. Hence $p_{(i)} \leqslant q_{(j)}$ by Lemma 1.1. For $i = j$ we have shown that $p_{(i)} \subseteq q_{(i)}$. If $i \neq j$, then $p_{(i)} = 0$ since $q_{(j)}$ is proper; hence $p_{(i)} \leqslant q_{(i)}$ follows trivially. We now induct on $\langle \varrho(p), \varrho(q) \rangle$. If $p = \bigwedge S$ and $p \subseteq q$ by $(\wedge \mathrm{W})$, then $t \subseteq q$ for some $t \in S$. Hence

$$p_{(i)} = \bigwedge (s_{(i)} \mid s \in S) \leqslant t_{(i)} \leqslant q_{(i)},$$

where the last inequality follows because $\langle \varrho(t), \varrho(q) \rangle$ is less than $\langle \varrho(p), \varrho(q) \rangle$. If $p = \bigvee S$ and $p \subseteq q$ by $(\vee \mathrm{W})$, then

$$p_{(i)} = \bigvee (s_{(i)} \mid i \in I) \leqslant q_{(i)}$$

since $s_{(i)} \leqslant q_{(i)}$ for all $s \in S$ by induction. The remaining two cases and the case of upper covers are similar.

(iii) We induct on $\langle \varrho(p), \varrho(q), \varrho(r) \rangle$ ordered lexicographically. (Any well-ordered extension of the product order would serve.) If $p \subseteq q$ holds by (C), $(\wedge \mathrm{W})$, or $(\vee \mathrm{W})$, or $q \subseteq r$ holds by (C), $(\mathrm{W} \wedge)$, or $(\mathrm{W} \vee)$, then the proof is routine; thus, we assume that $q = \bigvee S$, that $p \subseteq q$ holds by $(\mathrm{W} \vee)$, and that $q \subseteq r$ holds by $(\vee \mathrm{W})$. Hence $p \subseteq t$ for some $t \in S$ and $s \subseteq r$ for all $s \in S$. Therefore, $p \subseteq t \subseteq r$, and hence $p \subseteq r$ by induction. This completes the proof of Lemma 1.2.

By Lemma 1.2, $\subseteq$ is a quasi-ordering. Therefore, the relation $\equiv$ defined by

$$p \equiv q \quad \text{iff} \quad p \subseteq q \text{ and } q \subseteq p$$

is an equivalence relation. Further, $R(p) = \{q \mid p \equiv q\}$ is the equivalence class containing p. Therefore, $R(X) = \{R(p) \mid p \in P_{\mathfrak{m}}(X)\}$ is a poset with $R(p) \leqslant R(q)$ iff $p \subseteq q$.

LEMMA 1.3. *$R(X)$ is an $\mathfrak{m}$-lattice with*

$$\bigwedge \{R(s) \mid s \in S\} = R\left(\bigwedge S\right) \quad \textit{and} \quad \bigvee \{R(s) \mid s \in S\} = R\left(\bigvee S\right)$$

whenever $S \subseteq P_{\mathfrak{m}}(X)$ and $0 < |S| < \mathfrak{m}$. Furthermore, each L_i $(i \in I)$ is an $\mathfrak{m}$-sublattice of $R(X)$ in the obvious way. In other words, if $x = \inf Y$ in L_i with $x \in L_i$, $Y \subseteq L_i$, and $0 < |Y| < \mathfrak{m}$, then $R(x) = R(\bigwedge Y)$, and dually. Also, $R(x) \neq R(y)$ whenever $x \neq y$ in X.

Proof. If $p \subseteq s$ for all $s \in S$, then $p \subseteq \bigwedge S$ by $(\mathrm{W}\wedge)$. Since $\bigwedge S \subseteq s$ for all $s \in S$ by $(\wedge\mathrm{W})$, the first statement follows by duality. Let $x = \inf Y$ in L_i. Then $x \subseteq y$ for all $y \in Y$ and, therefore, $x \subseteq \bigwedge Y$. Using $(\bigwedge Y)^{(i)} = x$, we see that $\bigwedge Y \subseteq x$ holds by (C). Since $x \subseteq y$ can only hold by (C), an appeal to Lemma 1.1 completes the proof.

To complete the proof of Theorem 1.1, it remains to show that $R(X)$ is the free $\mathfrak{m}$-product of $(L_i \mid i \in I)$. Each L_i is an $\mathfrak{m}$-sublattice of $R(X)$ by Lemma 1.3 and $R(X)$ is clearly $\mathfrak{m}$-generated by X. Let K be an $\mathfrak{m}$-lattice and let the $\mathfrak{m}$-homomorphisms $\varphi_i \colon L_i \to K$ be given $(i \in I)$. We define $\psi \colon P_{\mathfrak{m}}(X) \to K$ inductively as follows: if $x \in L_i$, then $x\psi = x\varphi_i$; if $p = \bigwedge S$ and $s\psi$ is already defined for each $s \in S$, then

$$p\psi = \bigwedge (s\psi \mid s \in S);$$

if $p = \bigvee S$, then $p\psi$ is defined dually. We require the following

LEMMA 1.4. *Let $p, q \in P_{\mathfrak{m}}(X)$ and $i \in I$.*

(i) *If $p_{(i)}$ is proper, then $p_{(i)}\psi \leqslant p\psi$.*

(ii) *If $p^{(i)}$ is proper, then $p\psi \leqslant p^{(i)}\psi$.*

(iii) *$p \subseteq q$ implies that $p\psi \leqslant q\psi$.*

Proof. (i) If $p \in X$, then $p = p_{(i)}$. Hence $p_{(i)}\psi \leqslant p\psi$. If $p = \bigwedge S$, then

$$p_{(i)}\psi = \left(\bigwedge (s_{(i)} \mid s \in S)\right)\psi = \bigwedge (s_{(i)}\psi \mid s \in S) \leqslant \bigwedge (s\psi \mid s \in S) = p\psi.$$

(By induction, $s_{(i)}\psi \leqslant s\psi$ for all $s \in S$.) The calculation is similar if $p = \bigvee S$.

(ii) This is dual to (i).

(iii) If $p \subseteq q$ follows by (C), then $p^{(i)} \leqslant q_{(i)}$ for some $i \in I$. Applying (i) and (ii), we obtain

$$p \leqslant p^{(i)}\psi \leqslant q_{(i)}\psi \leqslant q\psi.$$

If $p \subseteq q$ holds by $(\wedge\mathrm{W})$ with $p = \bigwedge S$, then $s \subseteq q$ for some $s \in S$. Hence $p \leqslant s \leqslant q$. The remaining cases are analogous.

Thus ψ induces a map φ: $R(X) \to K$ that extends each φ_i. If $S \subseteq P_{\mathfrak{m}}(X)$ with $0 < |S| < \mathfrak{m}$, then

$$\big(\bigwedge(R(s) \mid s \in S)\big)\varphi = \big(R(\bigwedge S)\big)\varphi = (\bigwedge S)\psi$$
$$= \bigwedge(s\psi \mid s \in S) = \bigwedge\big(R(s)\varphi \mid s \in S\big).$$

We conclude that φ is an $\mathfrak{m}$-homomorphism, completing the proof of Theorem 1.1.

We shall reformulate the Structure Theorem in terms of elements of L. The upper and lower i-covers of elements of L have already been considered. In particular, $a_{(i)}$ is $a\varphi$, where φ: $L \to (L_i)^b$ is the $\mathfrak{m}$-homomorphism introduced before the statement of Theorem 1.1. Whenever $\mathfrak{m}$-lattices are considered in the sequel and we write $\bigwedge S$ or $\bigvee S$, it is to be understood that $0 < |S| < \mathfrak{m}$. Further, each L_i $(i \in I)$ is assumed to be an $\mathfrak{m}$-sublattice of L.

THEOREM 1.2. *Let the* $\mathfrak{m}$-*lattice* L *be the free* $\mathfrak{m}$-*product of the* $\mathfrak{m}$-*lattices* L_i, $i \in I$. *If* $a = \bigwedge S \leqslant \bigvee T = b$ *in* L, *then one of the following conditions holds*:

(C) *For some* $i \in I$, $a^{(i)} \leqslant b_{(i)}$.

$(_\wedge \mathrm{W})$ $s \leqslant b$ *for some* $s \in S$.

$(\mathrm{W}_\vee)$ $a \leqslant t$ *for some* $t \in T$.

Proof. As usual, let $X = \dot{\bigcup}(L_i \mid i \in I) \subseteq L$. For each $u \in S \cup T$, let $r_u \in P_{\mathfrak{m}}(X)$ represent u. We define the $\mathfrak{m}$-polynomials

$$p = \bigwedge(r_s \mid s \in S) \quad \text{and} \quad q = \bigvee(r_t \mid t \in T).$$

Then p and q represent a and b, respectively. Thus, by Theorem 1.1, $p \subseteq q$. If (C) holds for p and q in Theorem 1.1, then (C) holds here since $a^{(i)} = p^{(i)}$ and $b_{(i)} = q_{(i)}$ for $i \in I$. If $(_\wedge \mathrm{W})$ applies in Theorem 1.1, then $r_s \subseteq q$ for some $s \in S$, and hence $s \leqslant b$ in L. Finally, if $(\mathrm{W}_\vee)$ applies, a similar argument completes the proof.

COROLLARY 1.1. *Let* L *be the free* $\mathfrak{m}$-*product of* $(L_i \mid i \in I)$ *as above, with* $X = \dot{\bigcup}(L_i \mid i \in I)$. *No element of* $L - X$ *is both* $\mathfrak{m}$-*meet-reducibile and* $\mathfrak{m}$-*join-reducible. In other words, if* $a = \bigwedge S = \bigvee T$ *and* $a \notin S \cup T$, *then* $a \in X$.

Proof. Consider the inequality $\bigwedge S \leqslant \bigvee T$. If $(_\wedge \mathrm{W})$ holds, then $s \leqslant a$ for some $s \in S$ and, therefore, $s = a$, a contradiction. Similarly, $(\mathrm{W}_\vee)$ implies that $a \in T$. Thus, (C) must hold, implying that $a \in X$ because $a \leqslant a^{(i)} \leqslant a_{(i)} \leqslant a$. This completes the proof of the corollary.

If L is the completely free $\mathfrak{m}$-lattice $\mathfrak{m}$-generated by a poset X, then the Structure Theorem for L given by Crawley and Dean [2] is obtained by replacing (C) in Theorem 1.1 by

(C′) $p, q \in X$ and $p \leqslant q$.

Let $L_0 *_{\mathfrak{m}} L_1$ denote the free $\mathfrak{m}$-product of the $\mathfrak{m}$-lattices L_0 and L_1. (As usual, $\mathfrak{m}$ is omitted if $\mathfrak{m} = \aleph_0$.)

We now consider some results whose proofs differ slightly from the finitary case.

THEOREM 1.3 (The Splitting Theorem). *If* $L = L_0 *_{\mathfrak{m}} L_1$, *then*

$$L = (L_0] \cup [L_1).$$

Proof. Since the above union is an $\mathfrak{m}$-sublattice of L that contains the $\mathfrak{m}$-generating set $L_0 \cup L_1$, it must equal L. If the union were not disjoint, then $x \geqslant y$ for some $x \in L_0$ and $y \in L_1$, an impossibility.

COROLLARY 1.2. $L_0 *_{\mathfrak{m}} L_1$ *is the disjoint union of four sets: the smallest convex* $\mathfrak{m}$*-sublattices containing* L_0 *and* L_1, *respectively,* $(L_0] \cap (L_1]$, *and* $[L_0) \cap [L_1)$.

PROPOSITION 1.1. *Let* K_i *be a (possibly empty)* $\mathfrak{m}$*-sublattice of* L_i *for* $i \in I$ *and let* L *be the free* $\mathfrak{m}$*-product of* $(L_i \mid i \in I)$. *If* K *is the* $\mathfrak{m}$*-sublattice of* L $\mathfrak{m}$*-generated by* $\bigcup (K_i \mid i \in I)$, *then* K *is a free* $\mathfrak{m}$*-product of* $(K_i \mid i \in I,\ K_i \neq \emptyset)$.

Proof. If p and q are $\mathfrak{m}$-polynomials in $\bigcup (K_i \mid i \in I)$, then $p \subseteq q$ holds in the free $\mathfrak{m}$-product of $(K_i \mid i \in I)$ exactly when it holds in the free $\mathfrak{m}$-product of $(L_i \mid i \in I)$.

For $\mathfrak{m}$-lattices and a cardinal $\mathfrak{n} \leqslant \mathfrak{m}$, we define three properties that are familiar for the finitary case:

$(\mathrm{W}_{\mathfrak{n}})$ if $\bigwedge S \leqslant \bigvee T$ with $0 < |S|,\ |T| < \mathfrak{n}$, then

$$[\bigwedge S, \bigvee T] \cap (S \cup T) \neq \emptyset;$$

$(\mathrm{SD}^{\mathfrak{n}}_{\wedge})$ if $a = b \wedge s$ for all $s \in S$ with $0 < |S| < \mathfrak{n}$, then

$$a = b \wedge \bigvee S;$$

$(\mathrm{SD}^{\mathfrak{n}}_{\vee})$ if $a = b \vee s$ for all $s \in S$ with $0 < |S| < \mathfrak{n}$, then

$$a = b \vee \bigwedge S.$$

For the finitary case, each property is usually expressed with $\mathfrak{n} = 3$, which implies the corresponding property for $\mathfrak{n} = \aleph_0$. Observe that the conclusion of $(\mathrm{W}_{\mathfrak{m}})$ is the disjunction of the conditions $(\wedge \mathrm{W})$ and $(\mathrm{W} \vee)$ in Theorem 1.2. Jónsson [13] showed that the completely free $\mathfrak{m}$-lattice $\mathfrak{m}$-generated by a poset satisfies $(\mathrm{W}_{\mathfrak{m}})$. From this result (or the next theorem) it follows that any free $\mathfrak{m}$-lattice satisfies $(\mathrm{W}_{\mathfrak{m}})$.

A subset A of an $\mathfrak{m}$-lattice L *satisfies* $(\mathrm{W}_{\mathfrak{m}})$ if $(\mathrm{W}_{\mathfrak{m}})$ holds whenever A contains $S \cup T$. The subset A *satisfies one of the* (SD)*-conditions* if the condition holds whenever A contains b and S. For a subset A of L, the free $\mathfrak{m}$-product of $(L_i \mid i \in I)$, $A_{(i)}$ denotes the set $\{a_{(i)} \mid a \in A\}$. The next two results for the finitary case are due to Grätzer and Lakser [10].

THEOREM 1.4. *Let* L *be the free* $\mathfrak{m}$*-product of the* $\mathfrak{m}$*-lattices* L_i, $i \in I$. *Let* A_i, $i \in I$, *be a subset of* $(L_i)^b$ *satisfying* $(\mathrm{W}_{\mathfrak{m}})$. *Further, let* A *be a subset of* L *satisfying* $A_{(i)} \cup A^{(i)} \subseteq A_i$ *for all* $i \in I$. *Then* A *satisfies* $(\mathrm{W}_{\mathfrak{m}})$ *in* L.

Proof. Suppose that $a = \bigwedge S \leqslant \bigvee T = b$ with $S \cup T \subseteq A$. We shall apply Theorem 1.2. If $a^{(i)} \leqslant b_{(i)}$ for some $i \in I$, then the set $[a^{(i)}, b_{(i)}] \cap (S^{(i)} \cup T_{(i)})$ is nonempty. If, for example, $s^{(i)} \leqslant b$ with $s \in S$, then $s \in [a, b]$. Thus $(\mathrm{W}_{\mathfrak{m}})$ holds if (C) applies. If one of the remaining two conditions of Theorem 1.2 applies, $(\mathrm{W}_{\mathfrak{m}})$ obviously holds. Therefore, A satisfies $(\mathrm{W}_{\mathfrak{m}})$.

At this point, we have generalized all theorems of Chapter VI, Section 1, of [5] except for the last three theorems. The Normal Form Theorem of [8] replaces Theorem 6.1.16 of [5] which involves the concept of minimal polynomial.

THEOREM 1.5. *Let L be the free* $\mathfrak{m}$*-product of the* $\mathfrak{m}$*-lattices L_i, $i \in I$. Let A_i, $i \in I$, be a subset of $(L_i)^b$ satisfying* $(\mathrm{SD}^{\mathfrak{m}}_{\vee})$. *Further, let A be a subset of L satisfying $A_{(i)} \subseteq A_i$ for all $i \in I$. Then A satisfies* $(\mathrm{SD}^{\mathfrak{m}}_{\vee})$.

Proof. Let $a = b \vee s$ in L for each $s \in S$, where $0 < |S| < \mathfrak{m}$ and $\{b\} \cup S \subseteq A$. We can assume that $a \neq b$ and $a \notin S$. It follows that a is $\mathfrak{m}$-join-reducible. Hence, by Corollary 2 of [8], there is $T \subseteq L$ such that $a = \bigvee T$ and

(i) every element of $T - X$ is $\mathfrak{m}$-join-irreducible, where $X = \bigcup (L_i \mid i \in I) \subseteq L$;

(ii) if $t \in T - X$, then there is $U_t \subseteq L$ such that $t = \bigwedge U_t$ and $u \nleqslant a$ for all $u \in U_t$;

(iii) if $t \leqslant a_{(i)}$ for some $t \in T$ and $i \in I$, then $t = a_{(i)}$.

(If $a \in X$, then we can set $T = \{a\}$.) For each $s \in S$ and $t \in T$, consider the valid inequality $t \leqslant b \vee s$. If $t \in X$, then $t = a_{(i)}$ for some $i \in I$. Since $a_{(i)} = b_{(i)} \vee s_{(i)}$ for all $s \in S$, we conclude that $a_{(i)} = b_{(i)} \vee (\bigwedge S)_{(i)}$. If $t \notin X$, we consider $\bigwedge U_t \leqslant b \vee s$ with U_t as in (ii) above. Condition (C) would imply that $t \leqslant a_{(i)}$, yielding a contradiction by (iii). Also, $(_{\wedge}\mathrm{W})$ would contradict (ii). Thus $(\mathrm{W}_{\vee})$ holds. Consequently, $t \leqslant b$ or $t \leqslant s$. Therefore, for any $t \in T$, we obtain $t \leqslant b \vee \bigwedge S$. Thus $a = \bigvee T \leqslant b \vee \bigwedge S$, completing the proof.

The above proof is easily modified to yield a proof of the next theorem.

THEOREM 1.6. *If L is the completely free* $\mathfrak{m}$*-lattice* $\mathfrak{m}$*-generated by a poset, then L satisfies* $(\mathrm{SD}^{\mathfrak{m}}_{\vee})$ *and* $(\mathrm{SD}^{\mathfrak{m}}_{\wedge})$.

By Theorem 1.5 or Theorem 1.6, it follows that any free $\mathfrak{m}$-lattice satisfies $(\mathrm{SD}^{\mathfrak{m}}_{\vee})$ and $(\mathrm{SD}^{\mathfrak{m}}_{\wedge})$.

The next result for the finitary case is due to Grätzer and Sichler [12]. (The lower cover of x in A is denoted by x_A.)

THEOREM 1.7 (Common Refinement Property for Free $\mathfrak{m}$-Products). *Let L be a free* $\mathfrak{m}$*-product of A_0 and A_1 and also of B_0 and B_1. Theh L is a free* $\mathfrak{m}$*-product of*

$$(A_i \cap B_j \mid i, j = 0, 1,\ A_i \cap B_j \neq \emptyset).$$

Proof. As in [5], it suffices by Proposition 1.1 to show that L is $\mathfrak{m}$-generated by the union of the above intersections. Let $a \in A_0$. We shall show that $a_{B_0} \in A_0 \cup \{0\}$. Observe that L is $\mathfrak{m}$-generated by $B_0 \cup B_1$. For notational simplicity, assume that $a = p(b_0, b_1)$ with $b_i \in B_i$ ($i = 0, 1$). (To attain full generality, b_i should be viewed as a sequence of elements of B_i.) Then $a = p((b_0)_{A_0}, (b_1)_{A_0})$, and thus

$$a_{B_0} = p(b_0, 0) = p\big(((b_0)_{A_0})_{B_0}, ((b_1)_{A_0})_{B_0}\big) = p\big(((b_0)_{A_0})_{B_0}, 0\big),$$

where the last equality follows from $((b_1)_{A_0})_{B_0} \leqslant (b_1)_{B_0} = 0$. Since p is isotone, $a_{B_0} = p((b_0)_{A_0}, 0)$ is an element of $(A_0 \cap B_0) \cup \{0\}$. Thus $a = p((b_0)_{A_0}, (b_1)_{A_0})$ and A_0 is $\mathfrak{m}$-generated by $(A_0 \cap B_0) \cup (A_0 \cap B_1)$.

2. Intact and closure sublattices. Let A be a (possibly empty) subposet of B. We call A an *intact sublattice* of B if all existing joins and meets (of nonempty subsets) in A are preserved (remain intact) in B. (Neither A nor B need to be complete.) We form

$$B^b = B \cup \{0, 1\} \quad \text{and} \quad A^b = A \cup \{0, 1\}.$$

The element $b_A \in A^b$ is the *lower A-cover* of $b \in B^b$ if, for all $a \in A$, $a \leqslant b_A$ is equivalent to $a \leqslant b$. In particular, b_A is uniquely determined by b if it exists and $b_A \leqslant b$. *Upper A-covers* are defined dually. A is called a *closure sublattice* of B if each element of B has a lower and an upper A-cover. If A and C are $\mathfrak{m}$-lattices, then we know that A is a closure sublattice of $A *_{\mathfrak{m}} C$. This example motivates the definition and allows us to use the same terminology and notation for covers. Trivial examples of closure sublattices are $A = \emptyset$, $A = B$, and A a finite chain. Observe that being a closure sublattice is a transitive property.

If A is a closure sublattice of B, then A is an intact sublattice of B. Indeed, if $\emptyset \neq S \subseteq A$, $c = \sup_A S$, $b \in B$ is an upper bound of S, and b_A exists, then

$$c = \sup_A S \leqslant \sup_A ((b] \cap A) = b_A \leqslant b.$$

Let $\emptyset \neq S \subseteq A$ and suppose that A is a closure sublattice of B. If $b = \sup_B A$, then

$$b = b_A = \sup_A S.$$

Thus, a closure sublattice of an $\mathfrak{m}$-lattice is $\mathfrak{m}$-complete. Also, observe that if u is the unit of B and A is a closure sublattice of B, then u_A is the unit of A.

The next two theorems show that closure sublattices occur naturally in both free $\mathfrak{m}$-products and completely free $\mathfrak{m}$-lattices. In the proofs that follow, $P^0_{\mathfrak{m}}(X)$ will denote the result of adding a new element 0 to $P_{\mathfrak{m}}(X)$. Also, in this context, $\bigvee \emptyset = 0$.

THEOREM 2.1. *Let L_i, $i \in I$, be* $\mathfrak{m}$*-lattices, let K_i be a (possibly empty) closure sublattice of L_i for $i \in I$, and let L be the free* $\mathfrak{m}$*-product of $(L_i \mid i \in I)$. If K is the* $\mathfrak{m}$*-sublattice of L* $\mathfrak{m}$*-generated by $\bigcup(K_i \mid i \in I)$, then K is a closure sublattice of L.*

Proof. Let $X = \dot{\bigcup}(L_i \mid i \in I)$ and $Y = \dot{\bigcup}(K_i \mid i \in I)$. Since $Y \subseteq X$, we have $P^0_{\mathfrak{m}}(Y) \subseteq P^0_{\mathfrak{m}}(X)$. For $q \in P^0_{\mathfrak{m}}(X)$, we inductively define $q' \in P^0_{\mathfrak{m}}(Y)$ as follows:

(i) $0' = 0$;

(ii) $x' = x_{K_i}$ if $x \in L_i$;

(iii) if $q = \bigvee T$, then $q' = \bigvee(t' \mid t' \neq 0, t \in T)$;

(iv) if $q = \bigwedge T$, then

$$q' = \begin{cases} \bigwedge(t' \mid t \in T) & \text{if } t' \neq 0 \text{ for all } t \in T, \\ 0 & \text{otherwise.} \end{cases}$$

If $b \in L$ is represented by $q \in P_{\mathfrak{m}}(X)$, we shall show that q' represents b_K in L. (Hence, by duality, K is a closure sublattice of L.) Since $q' \in P^0_{\mathfrak{m}}(Y)$, it represents an element of $K \cup \{0\}$. It is easily shown that $q' \subseteq q$ for all $q \in P^0_{\mathfrak{m}}(X)$. (The relation $\subseteq$ on $P^0_{\mathfrak{m}}(X)$ is for the free product L.) Let $x \in L_i$ and $x \subseteq q$ with $i \in I$ and $q \in P_{\mathfrak{m}}(X)$. By induction on the rank of q, it is easily shown that $x_{K_i} \subseteq q'$.

Let $p \in P_{\mathfrak{m}}(Y)$, $q \in P_{\mathfrak{m}}(X)$, and $p \subseteq q$. We show, by induction on the ranks of p and q, that $p \subseteq q'$. If (C) applies to $p \subseteq q$, then $p^{(i)} \leqslant q_{(i)}$ in L_i for some $i \in I$. Since $q_{(i)} \subseteq q$, $(q_{(i)})_{K_i}$, we have $p \subseteq q'$. The cases $p = \bigvee S$ (respectively, $p = \bigwedge S$) and $s \subseteq q$ for all (respectively, for some) $s \in S$ are left to the reader. If $q = \bigvee T$ and $p \subseteq t$ for some $t \in T$, then $p \subseteq t'$ by induction. Thus,

$$p \subseteq \bigvee(t' \mid t' \neq 0, t \in T) = q'.$$

The final case, $q = \bigwedge T$, is similar.

THEOREM 2.2. *Let Y be a subset of an* $\mathfrak{m}$*-lattice L such that $0 < |Y| < \mathfrak{m}$. We consider two conditions on L and Y:*

(α) *L is the free* $\mathfrak{m}$*-product of a family $(L_i \mid i \in I)$ of* $\mathfrak{m}$*-lattices, $X = \dot{\bigcup}(L_i \mid i \in I) \subseteq L$, and each element of Y is incomparable with each element of X. (Equivalently, the i-covers of each element of Y are improper for all $i \in I$.)*

(β) *L is the completely free* $\mathfrak{m}$*-lattice* $\mathfrak{m}$*-generated by a poset X.*

If L and Y satisfy (α) or (β) and K is the $\mathfrak{m}$*-sublattice of L* $\mathfrak{m}$*-generated by Y, then K is a closure sublattice of L.*

Proof. Choose a map from Y to $P_{\mathfrak{m}}(X)$ so that each element of Y is represented in L by its image under this map. Let φ: $P^0_{\mathfrak{m}}(Y) \to P^0_{\mathfrak{m}}(X)$ be the obvious extension of this map satisfying $0\varphi = 0$. Let $q \in P^0_{\mathfrak{m}}(X)$. We set

$$q^* = \bigvee(y \in Y \mid y\varphi \subseteq q).$$

Since $|Y| < \mathfrak{m}$, we have $q^* \in P^0_{\mathfrak{m}}(Y)$. If we assume ($\alpha$), then $x^* = 0$ whenever $x \in X$. We inductively define $q' \in P^0_{\mathfrak{m}}(Y)$ as follows:

(i) $0' = 0$;

(ii) $x' = x^*$ if $x \in X$;

(iii) if $q = \bigvee T$, then

$$q' = \begin{cases} q^* \vee \bigvee(t' \mid t' \neq 0,\ t \in T) & \text{if } t' \neq 0 \text{ for some } t \in T, \\ q^* & \text{otherwise;} \end{cases}$$

(iv) if $q = \bigwedge T$, then

$$q' = \begin{cases} \bigwedge(t' \mid t \in T) & \text{if } t' \neq 0 \text{ for all } t \in T, \\ 0 & \text{otherwise.} \end{cases}$$

If $b \in L$ is represented by $q \in P_{\mathfrak{m}}(X)$, we shall show that $q'\varphi$ represents b_K in L. (Hence, by duality, K is a closure sublattice of L.) Clearly, $q'\varphi$ represents an element of K^b. It is easily shown that $q^*\varphi \subseteq q'\varphi \subseteq q$ for all $q \in P^0_{\mathfrak{m}}(X)$.

Let $p \in P_{\mathfrak{m}}(Y)$, $q \in P_{\mathfrak{m}}(X)$, and $p\varphi \subseteq q$. We show, by induction on the ranks of p and q, that $p\varphi \subseteq q'\varphi$. If $p \in Y$, then $p\varphi \subseteq q^*\varphi \subseteq q'\varphi$. In case ($\alpha$), $(p\varphi)^{(i)} = 1$ for all $i \in I$ and, consequently, (C) cannot apply to $p\varphi \subseteq q$. (In particular, $q \notin X$.) Next, suppose (β) holds and $q \in X$. If $p = \bigvee S$ (respectively, $p = \bigwedge S$), then $s\varphi \subseteq q$ for all (respectively, for some) $s \in S$. The relation $p\varphi \subseteq q\varphi$ now follows easily by induction. Henceforth, we assume that either (α) or (β) holds. If p is a join or q is a meet, the relation $p\varphi \subseteq q'\varphi$ is easily derived. We can assume that $p \notin Y$, $p = \bigwedge S$, $q = \bigvee T$, and either ($_\wedge$W) or (W$_\vee$) applies to $p\varphi \subseteq q$. In the first case, $s\varphi \subseteq q$ for some $s \in S$. Hence $p\varphi \subseteq s\varphi \subseteq q'\varphi$. Otherwise, $p\varphi \subseteq t$ for some $t \in T$ and, consequently, $p\varphi \subseteq t'\varphi \subseteq q'\varphi$. This completes the proof of Theorem 2.2.

COROLLARY 2.1. *If Y is any poset with $0 < |Y| < \mathfrak{m}$, then $CF_{\mathfrak{m}}(Y)$ is isomorphic to a closure sublattice of $F_{\mathfrak{m}}(3)$.*

Proof. Crawley and Dean [2] have shown that $CF_{\mathfrak{m}}(Y)$ is isomorphic to an $\mathfrak{m}$-sublattice K of $F_{\mathfrak{m}}(3)$. By Theorem 2.2 (β), K is a closure sublattice of $F_{\mathfrak{m}}(3)$.

Remarks. 1. By Proposition 1.1, the $\mathfrak{m}$-lattice K in Theoerem 2.1 is a free $\mathfrak{m}$-product of $(K_i \mid i \in I,\ K_i \neq \emptyset)$.

2. It follows from the proof that the $\mathfrak{m}$-lattice K in Theorem 2.2 must satisfy (W$_{\mathfrak{m}}$). (Of course, in (β), this is obvious since L satisfies (W$_{\mathfrak{m}}$) by [13].)

3. The restriction $|Y| < \mathfrak{m}$ is necessary in Corollary 2.1: this statement may fail for $|Y| = \mathfrak{m}$. For example, $K = F_{\mathfrak{m}}(\mathfrak{m})$ is an $\mathfrak{m}$-sublattice of $L = F_{\mathfrak{m}}(3)$ by [2], but it follows by an observation made before Theorem 2.1 that K cannot be a closure sublattice of L because K has no unit.

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