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ON WEAK AGASSIZ SYSTEMS OF ALGEBRAS

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In [1] we dealt with the concept of normal Agassiz system of algebras being a modification of Agassiz system introduced by Grätzer and Sichler in [2]. Now we discuss a generalization of the concept of normal Agassiz system in the direction suggested by the definition of a naming functor given by Grätzer and Sichler in [2]. The present paper is a continuation of [1], and thus all notational and terminological conventions of [1] will be used freely. For the sake of convenience, in some situations we shall use the notation $\boldsymbol{p} \equiv_K \boldsymbol{q}$ interchangeably with $\boldsymbol{p} \equiv \boldsymbol{q} \in \mathrm{Id}(K)$. In several cases, to make evident the length of a sequence of polynomial symbols we shall use the notation $(\boldsymbol{p}, \ldots, \boldsymbol{p})^k$ for a k-termed ($k \geqslant 1$) sequence of $\boldsymbol{p}$'s.

Given a naming functor $N\colon \boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$ (see [1]), we say that an algebra $\mathfrak{B}$ of type ϱ belongs to the *weak structurality class* of N ($\mathfrak{B} \in \mathrm{WSC}(N)$) if, for every n-ary ($n \geqslant 1$) polynomial symbol $\boldsymbol{p} \in \boldsymbol{P}(\tau)$ and for every $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n \in \boldsymbol{P}(\tau)$,

$$N\big(\boldsymbol{p}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)\big) \equiv N(\boldsymbol{p})\big(N(\boldsymbol{q}_1), \ldots, N(\boldsymbol{q}_n)\big) \in \mathrm{Id}(\mathfrak{B}).$$

Obviously, $\mathrm{WSC}(N)$ always is an equational class and $\mathrm{SC}(N) \subseteq \mathrm{WSC}(N)$. Note also that in order to assure the closedness of the set of identities $\mathrm{Id}_N(I)$ it is sufficient to assume that $I \subseteq \mathrm{WSC}(N)$ (cf. [1]). Although the class $\mathrm{WSC}(N)$ is defined simply by dropping condition (ii) in the definition of the class $\mathrm{SC}(N)$ in [1], we have the following

LEMMA 1. *If $I \subseteq \mathrm{WSC}(N)$ and there exists a variable $\boldsymbol{x}_k$ such that $N(\boldsymbol{x}_k) \equiv \boldsymbol{x}_k \in \mathrm{Id}(I)$, then $I \subseteq \mathrm{SC}(N)$.*

Proof. Suppose that $I \subseteq \mathrm{WSC}(N)$, $N(\boldsymbol{x}_k) \equiv_I \boldsymbol{x}_k$ and $\boldsymbol{x}_i$ is an arbitrary variable. Then

$$\begin{aligned} N(\boldsymbol{x}_i) &= \big(N(\boldsymbol{x}_i)(\boldsymbol{x}_k, \ldots, \boldsymbol{x}_k)^i\big)(\boldsymbol{x}_i, \ldots, \boldsymbol{x}_i)^k \\ &\equiv_I \big(N(\boldsymbol{x}_i)(N(\boldsymbol{x}_k), \ldots, N(\boldsymbol{x}_k))^i\big)(\boldsymbol{x}_i, \ldots, \boldsymbol{x}_i)^k \\ &\equiv_I N\big(\boldsymbol{x}_i(\boldsymbol{x}_k, \ldots, \boldsymbol{x}_k)^i\big)(\boldsymbol{x}_i, \ldots, \boldsymbol{x}_i)^k \\ &= N(\boldsymbol{x}_k)(\boldsymbol{x}_i, \ldots, \boldsymbol{x}_i)^k \equiv_I \boldsymbol{x}_k(\boldsymbol{x}_i, \ldots, \boldsymbol{x}_i)^k = \boldsymbol{x}_i, \end{aligned}$$

which proves that $I \subseteq \mathrm{SC}(N)$.

A naming functor $N\colon \boldsymbol{P}(\tau)\to\boldsymbol{P}(\varrho)$ will be called *standard* if, for every n-ary ($n\geqslant 1$) polynomial symbol $\boldsymbol{p}\in\boldsymbol{P}(\tau)$ and for every $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n\in\boldsymbol{P}(\tau)$,

$$N\big(\boldsymbol{p}(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)\big) = N(\boldsymbol{p})\big(N(\boldsymbol{q}_1),\ldots,N(\boldsymbol{q}_n)\big).$$

LEMMA 2. *If a naming functor N is standard, then* SC(N) *is the class of all algebras of type ϱ.*

Proof. It suffices to show that if N is standard, then $N(\boldsymbol{x}_i)=\boldsymbol{x}_i$ for every variable $\boldsymbol{x}_i$. Indeed, suppose that $\boldsymbol{x}_k$ is a variable such that $N(\boldsymbol{x}_k)\neq\boldsymbol{x}_k$. Then $N(\boldsymbol{x}_k)$ contains some operation symbol and, therefore,

$$N(\boldsymbol{x}_k) = N\big(\boldsymbol{x}_k(\boldsymbol{x}_k,\ldots,\boldsymbol{x}_k)^k\big)\neq N(\boldsymbol{x}_k)\big(N(\boldsymbol{x}_k),\ldots,N(\boldsymbol{x}_k)\big)^k,$$

since operation symbols occur in $N(\boldsymbol{x}_k)\big(N(\boldsymbol{x}_k),\ldots,N(\boldsymbol{x}_k)\big)^k$ more times than in $N(\boldsymbol{x}_k)$. This completes the proof.

With each naming functor $N\colon\boldsymbol{P}(\tau)\to\boldsymbol{P}(\varrho)$ we associate a mapping $\bar{N}\colon\boldsymbol{P}(\tau)\to\boldsymbol{P}(\varrho)$ defined as follows:

(i) $\bar{N}(\boldsymbol{x}_i)=\boldsymbol{x}_i$ for every variable $\boldsymbol{x}_i$;

(ii) $\bar{N}(\boldsymbol{f})=N(\boldsymbol{f})$ if $\boldsymbol{f}$ is a nullary operation symbol of type τ;

(iii) $\bar{N}\big(\boldsymbol{f}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)\big)=N\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big)\big(\bar{N}(\boldsymbol{p}_1),\ldots,\bar{N}(\boldsymbol{p}_n)\big)$ if $\boldsymbol{f}$ is an n-ary ($n\geqslant 1$) operation symbol of type τ and $\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\in\boldsymbol{P}(\tau)$.

LEMMA 3. *If $N\colon\boldsymbol{P}(\tau)\to\boldsymbol{P}(\varrho)$ is a naming functor, then the following conditions hold*:

(i) *$\bar{N}$ is a standard naming functor*;

(ii) *if $I\subseteq$* WSC(N), *then $N(\boldsymbol{p})\equiv_I\bar{N}(\boldsymbol{p})\big(N(\boldsymbol{x}_1),\ldots,N(\boldsymbol{x}_n)\big)$ for every n-ary ($n\geqslant 1$) polynomial symbol $\boldsymbol{p}\in\boldsymbol{P}(\tau)$*;

(iii) *if $I\subseteq$* SC(N), *then $N(\boldsymbol{p})\equiv_I\bar{N}(\boldsymbol{p})$ for every $\boldsymbol{p}\in\boldsymbol{P}(\tau)$*;

(iv) *$N\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big)=\bar{N}\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big)$ for every n-ary ($n\geqslant 1$) operation symbol $\boldsymbol{f}$ of type τ*;

(v) *if $I\subseteq$* WSC(N) *and $\boldsymbol{x}_k$ is a variable, then, for every polynomial symbol $\boldsymbol{p}\in\boldsymbol{P}(\tau)$ being not a variable,*

$$N(\boldsymbol{x}_k)\big(\bar{N}(\boldsymbol{p}),\ldots,\bar{N}(\boldsymbol{p})\big)^k\equiv_I\bar{N}(\boldsymbol{p}).$$

Proof. Conditions (i), (ii), and (iii) can be proved by a standard induction argument on the rank of a polynomial symbol of type τ, and condition (iv) is obvious. To prove (v) observe first that it holds if $\boldsymbol{p}$ is a nullary operation symbol of type τ. Now let us suppose that $\boldsymbol{p}=\boldsymbol{f}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)$, where $\boldsymbol{f}$ is an n-ary ($n\geqslant 1$) operation symbol of type τ and $\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n\in\boldsymbol{P}(\tau)$. Then we get

$$\begin{aligned}
N(\boldsymbol{x}_k)\big(\bar{N}(\boldsymbol{p}),\ldots,\bar{N}(\boldsymbol{p})\big)^k &= N(\boldsymbol{x}_k)\Big(\bar{N}\big(\boldsymbol{f}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)\big),\ldots,\bar{N}\big(\boldsymbol{f}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)\big)\Big)^k\\
&= N(\boldsymbol{x}_k)\Big(N\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big)\big(\bar{N}(\boldsymbol{p}_1),\ldots,\bar{N}(\boldsymbol{p}_n)\big),\ldots,\\
&\qquad\qquad N\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big)\big(\bar{N}(\boldsymbol{p}_1),\ldots,\bar{N}(\boldsymbol{p}_n)\big)\Big)^k\\
&= \Big(N(\boldsymbol{x}_k)\Big(N\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big),\ldots,N\big(\boldsymbol{f}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\big)\Big)^k\Big)\big(\bar{N}(\boldsymbol{p}_1),\ldots,\bar{N}(\boldsymbol{p}_n)\big)
\end{aligned}$$

$$\equiv_I N\big(x_k(f(x_1, \ldots, x_n), \ldots, f(x_1, \ldots, x_n))^k\big)\big(\bar{N}(p_1), \ldots, \bar{N}(p_n)\big)$$
$$= N\big(f(x_1, \ldots, x_n)\big)\big(\bar{N}(p_1), \ldots, \bar{N}(p_n)\big) = \bar{N}\big(f(p_1, \ldots, p_n)\big) = \bar{N}(p),$$

which completes the proof.

Let us define a *weak Agassiz system* of algebras by replacing condition (iii) of the definition of a normal Agassiz system (see [1]) by the following:

(iii)$_w$ N: $\boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$ is a naming functor such that $\mathfrak{B} \in \mathrm{WSC}(N)$.

The *sum* of a weak Agassiz system will be defined exactly as that of a normal Agassiz system and the notation $[I, K, N]$ and $\lim[I, K, N]$ will be used for weak Agassiz systems in the same manner as (I, K, N) and $\lim(I, K, N)$ for normal Agassiz systems in [1].

From now on, it will always be assumed that K and I are non-empty classes of algebras of types τ and ϱ, respectively, and that N: $\boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$ is a naming functor such that $I \subseteq \mathrm{WSC}(N)$.

THEOREM 1. (i) $\lim[I, K, N] = \lim(I, K, \bar{N})$.

(ii) $\mathrm{Sm}\big(\mathrm{Id}_{\bar{N}}(I) \cap \mathrm{Id}(K)\big) \subseteq \mathrm{Id}(\lim[I, K, N]) \subseteq \mathrm{Id}_{\bar{N}}(I) \cap \mathrm{Id}(K)$.

Proof. In view of Lemma 3 (i), (iv), it is obvious that a weak Agassiz system

$$\big(\mathfrak{B}, (\mathfrak{A}_b \mid b \in B), (h_{bc} \mid \langle b, c\rangle \in R), N\big)$$

belongs to $[I, K, N]$ if and only if the normal Agassiz system

$$\big(\mathfrak{B}, (\mathfrak{A}_b \mid b \in B), (h_{bc} \mid \langle b, c\rangle \in R), \bar{N}\big)$$

belongs to $(I, K, \bar{N})$, which proves (i). Condition (ii) follows immediately from (i) and Proposition 1 of [1].

THEOREM 2. *If* $I \nsubseteq \mathrm{SC}(N)$, *then every identity in* $\mathrm{Id}_{\bar{N}}(I)$ *is symmetric and*

$$\mathrm{Id}(\lim[I, K, N]) = \mathrm{Sm}\big(\mathrm{Id}_{\bar{N}}(I) \cap \mathrm{Id}(K)\big).$$

Proof. Let us suppose that, for some variable x_k and some polynomial symbol $p \in \boldsymbol{P}(\tau)$ being not a variable, $\bar{N}(p) \equiv_I \bar{N}(x_k)$. Then $\bar{N}(p) \equiv_I x_k$ and by Lemma 3 (v) we get

$$N(x_k) = N(x_k)(x_k, \ldots, x_k)^k \equiv_I N(x_k)\big(\bar{N}(p), \ldots, \bar{N}(p)\big)^k \equiv_I \bar{N}(p) \equiv_I x_k,$$

which gives $I \subseteq \mathrm{SC}(N)$ by virtue of Lemma 1. Thus we have proved that $I \nsubseteq \mathrm{SC}(N)$ implies that all identities in $\mathrm{Id}_{\bar{N}}(I)$ are symmetric which, by Theorem 1 (ii), yields the required equality.

Remark. If $I \subseteq \mathrm{SC}(N)$, then from Lemma 3 (iii) it follows that $\mathrm{Id}_{\bar{N}}(I) = \mathrm{Id}_N(I)$. If $I \nsubseteq \mathrm{SC}(N)$, then $\mathrm{Id}_{\bar{N}}(I) \subseteq \mathrm{Id}_N(I)$ by Lemma 3 (ii), but the converse inclusion does not hold in general. A suitable example shows that even symmetric identities of $\mathrm{Id}_N(I)$ need not belong to $\mathrm{Id}_{\overline{N}}(I)$.

REFERENCES

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