COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

EXTENSIONS OF CONTINUOUS FUNCTIONS INTO LOCALLY CONVEX SPACES OVER NON-ARCHIMEDEAN FIELDS

BY

ROBERT L. ELLIS (COLLEGE PARK, MARYLAND)

Arens [1], Theorem 4.1, proved a theorem concerning the extension of continuous functions from a closed subset of a paracompact space X into a complete metrizable convex subset of a locally convex space E. In this paper* the same problem is considered when E is replaced by a locally K-convex space [3] over a field K with a non-Archimedean absolute value and X is replaced by an ultraparacompact space (that is, one having the property that every open cover is refined by a locally finite cover consisting of sets which are both open and closed).

THEOREM. *Let A be a closed subset of an ultraparacompact space X and let E be a locally K-convex space over a field K with a non-Archimedean absolute value. Let C be a complete metrizable convex subset of E and $f\colon A \to C$ a continuous function. Then there is a continuous extension $F\colon X \to C$.*

Proof. We may assume that $0 \in C$. Let d be a metric for C and for any $\varepsilon > 0$ denote by $B(0, \varepsilon)$ the "open ball" $\{x \in C \mid d(x, 0) < \varepsilon\}$. By induction we will define a sequence $\{C_n \mid n \geqslant 1\}$ of K-convex neighborhoods of 0 in E such that $C_n \cap C \subseteq B(0, 1/n)$, and a sequence $\{g_n\colon X \to C_n \cap C \mid n \geqslant 1\}$ of continuous functions such that if f_n is the restriction of g_n to A, then the range of $f - \sum_{i=1}^{n} f_i$ is contained in $C_n \cap C$.

Let C_1 be a K-convex neighborhood of 0 in E such that $C_1 \cap C \subseteq B(0, 1)$. Since C_1 is a subgroup of E, it is both open and closed. Let $\{c_i \mid i \in I\}$ be a complete set of coset representatives of $C_1 \cap C$ in C and for each $i \in I$, let $A_i = f^{-1}(c_i + C_1 \cap C)$. Then $\{A_i \mid i \in I\}$ is an open partition of A. For each $i \in I$ let O_i be an open set in X such that $O_i \cap A = A_i$. Then $\{O_i \mid i \in I\} \cup \{cA\}$ is an open cover of X. Let $\mathscr{G} = \{G_j \mid j \in J\}$ be a locally finite refinement consisting of sets which are both open and closed. Then for each $j \in J$, either $G_j \subseteq cA$ or $G_j \subseteq O_i$ for some $i \in I$. If $G_j \subseteq cA$, let $c_j' = 0$. If $G_j \cap A \neq \emptyset$ and $G_j \subseteq O_i$, then $G_j \cap A \subseteq A_i$ and so G_j cannot

* Research supported by the National Science Foundation under Grant GP-7426.

be contained in O_k for $k \neq i$ since $\{A_k | k \in I\}$ is a partition of A. In this case let $c'_j = c_i$. For each $j \in J$ let χ_j be the characteristic function of G_j with values in K. Then each χ_j is continuous. Define $g_1\colon X \to C_1 \cap C$ by

$$g_1(x) = \sum_{j \in J} \chi_j(x) c'_j.$$

Since $\mathscr{G}$ is locally finite, this is meaningful and g_1 is continuous. Let f_1 be the restriction of g_1 to A. If $x \in A$ and $i \in I$ is such that $x \in A_i$, then $f(x) \in c_i + C_1 \cap C$ and $f_1(x) = c_i$ and so $f(x) - f_1(x) \in C_1 \cap C$.

Now assume that $n > 1$ and that $C_1, C_2, \ldots, C_{n-1}$ and $g_1, g_2, \ldots, g_{n-1}$ have been defined with the desired properties, and for $1 \leqslant i \leqslant n-1$ let f_i be the restriction of g_i to A. Repeat the above procedure with C_1 replaced by a K-convex neighborhood C_n of 0 in E such that $C_n \cap C \subseteq B(0, 1/n)$ and with f replaced by $f - \sum_{i=1}^{n-1} f_i$ to obtain a continuous function $g_n\colon X \to C_n \cap C$ such that if f_n is the restriction of g_n to A, then the range of $f - \sum_{i=1}^{n} f_i$ is contained in $C \cap C_n$. Since C is complete and the range of g_n is contained in $C_n \cap C \subseteq B(0, 1/n)$, it is clear that

$$F = \sum_{i=1}^{\infty} g_i$$

converges uniformly and hence is continuous. Since

$$\left(f - \sum_{i=1}^{n} f_i\right)(X) \subseteq C_n \cap C,$$

it follows that F extends f.

Some extension theorems for continuous functions into more general (non-linear) topological spaces will appear in another paper [2].

REFERENCES

[1] R. Arens, *Extensions of functions on fully normal spaces*, Pacific Journal of Mathematics 2 (1952), p. 11–22.

[2] R. L. Ellis, *Extending continuous functions on zero-dimensional spaces* (to appear).

[3] A. F. Monna, *Espaces localement convexes sur un corps valué*, Indagationes Mathematicae 24 (1962), p. 351–367.

Reçu par la Rédaction le 24. 11. 1969