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ON THE SET WHERE AN APPROXIMATE DERIVATIVE IS A DERIVATIVE

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Let f be a real-valued function defined on an interval and let E be the set of points x where the ordinary derivative $f'(x)$ exists. The purpose of this paper is to prove the following

THEOREM. *Assume that f has an approximate derivative $f'_{ap}(x)$ (finite or infinite) at every x and satisfies the condition that at each x at which* $\lim_{t\to x-} f(t)$ *or* $\lim_{t\to x+} f(t)$ *exists the limit is equal to $f(x)$. If λ is a real number such that $\{x\colon f'_{ap}(x) = \lambda\} \neq \emptyset$, then*

$$\{x\colon f'_{ap}(x) = \lambda\} \cap E \neq \emptyset.$$

This theorem seems slightly stronger than that obtained by O'Malley [3]. However, the lack of approximate continuity makes the proof much more complicated.

The concepts that we use in this paper can be found in [5]. To prove our theorem, we need the following lemmas which are consequences of Theorems 3, 4, 6 in [4] and Theorem 2 in [1].

LEMMA 1. *f'_{ap} is a Darboux function of Baire class* 1.

LEMMA 2. *If f'_{ap} is bounded from above or from below on an interval, then $f'_{ap} = f'$ there.*

Proof of the Theorem. For simplicity, noting that the function $f(x) - \lambda x$ satisfies all conditions set for f, we assume without loss of generality that f is defined on $[0, 1]$ and $\lambda = 0$.

Let F denote the set of points x such that either f'_{ap} is unbounded both from above and below on $[x, x+\delta)$ for every $\delta > 0$ or f'_{ap} is unbounded both from above and below on $(x-\delta, x]$ for every $\delta > 0$ and let $G = [0, 1] - F$. Clearly, G is open in $[0, 1]$ and F is closed. Moreover, owing to Lemma 2 and the assumption that $f'_{ap}(x)$ exists at every x, we see that $G \subset E$. If $F = \emptyset$, then $E = G = [0, 1]$ and there is nothing to prove. If $F \neq \emptyset$ and F has an isolated point x_0, then there exists a $\delta > 0$ such that $(x_0-\delta, x_0+\delta)\cap F$ is the single point x_0 and f'_{ap} is unbounded both from above and below on either $[x_0, x_0+\delta)$ or $(x_0-\delta, x_0]$. In view of Lemma 1, f'_{ap} takes all real values on either $(x_0, x_0+\delta)$ or $(x_0-\delta, x_0)$. It follows that there exists a point

$$x \in \big((x_0-\delta, x_0)\cup(x_0, x_0+\delta)\big)\cap[0,1] \subset G \subset E$$

such that $f'_{ap}(x) = 0$ and this is what we want. Now, we assume that F is a perfect set. By Lemma 1, f'_{ap} is a function of Baire class 1, and hence there exists a point ξ_0 in $F\cap(0,1)$ such that

$$\lim_{x\to\xi_0, x\in F} f'_{ap}(x) = f'_{ap}(\xi_0)$$

(finite or infinite).

Case 1. $f'_{ap}(\xi_0) > 0$. There exists a $\delta > 0$ such that $f'_{ap}(x) > 0$ whenever $|x-\xi_0| \leqslant \delta$ and $x \in F$. On the other hand, since $\xi_0 \in F$, there exists a ξ_1 such that $|\xi_1-\xi_0| \leqslant \delta$ and $f'_{ap}(\xi_1) < 0$. Thus $\xi_1 \in G$ and ξ_1 is contained in some component C of G. This component C is an open (relative to $[0,1]$) interval with one endpoint lying between ξ_0 and ξ_1 or equal to the point ξ_0. We denote this endpoint by ξ_2 and find that $|\xi_2-\xi_0| < \delta$ and $\xi_2 \in F$. Therefore, we have $f'_{ap}(\xi_1) < 0 < f'_{ap}(\xi_2)$. Since f'_{ap} is a Darboux function, there exists a ξ_3 lying between ξ_1 and ξ_2 such that $f'_{ap}(\xi_3) = 0$. Clearly, $\xi_3 \in C \subset G \subset E$. It follows that

$$\{x\colon f'_{ap}(x) = 0\}\cap E \neq \emptyset.$$

Case 2. $f'_{ap}(\xi_0) < 0$. The proof is similar to that in Case 1.

Case 3. $f'_{ap}(\xi_0) = 0$. There exists a $\delta > 0$ such that $|f'_{ap}(x)| < 1$ whenever $|x-\xi_0| \leqslant \delta$ and $x \in F$. Let

$$I_0 = [0,1]\cap[\xi_0-\delta, \xi_0+\delta] \quad \text{and} \quad Q = \{x \in I_0\colon f'_{ap}(x) = 0\}.$$

If $Q\cap G \neq \emptyset$, then $Q\cap E \neq \emptyset$ and our conclusion of the theorem follows. Suppose that $Q\cap G = \emptyset$; then $Q \subset F$. Since also $Q \subset I_0$ and $I_0\cap F$ is a closed set, we see that $\bar{Q} \subset I_0\cap F$, $\bar{Q}$ being the closure of Q. Therefore, $x \in \bar{Q}$ implies $|f'_{ap}(x)| < 1$. Let

$$K = \{x \in I_0\colon f'_{ap}(x) \text{ is finite}\};$$

we have $\bar{Q} \subset K$. Clearly, $\xi_0 \in Q\cap I_0^\circ$, where I_0° denotes the interior of I_0. We may choose a smaller I_0, if necessary, so that the endpoints of I_0 are not isolated points of Q. If Q has an isolated point x_i, then $x_i \in I_0^\circ$ and there exists a $\delta > 0$ such that

$$(x_i-\delta, x_i)\cup(x_i, x_i+\delta) \subset I_0 - Q.$$

By Lemma 1, we have either $f'_{ap} > 0$ on $(x_i-\delta, x_i)$ or $f'_{ap} < 0$ on $(x_i-\delta, x_i)$, and hence f'_{ap} is either bounded from below or bounded from above on $(x_i-\delta, x_i]$. By Lemma 2, $f'_{ap} = f'$ there, in particular, $f^-(x_i) = f'_{ap}(x_i) = 0$, where $f^-(x_i)$ is the left-hand ordinary derivative at x_i.

Similarly, $f^+(x_i)$ being the right-hand ordinary derivative at x_i, we have $f^+(x_i) = f'_{\mathrm{ap}}(x_i) = 0$. Thus there exists a derivative $f'(x_i) = 0$. It remains to show the case where Q has no isolated point or, in other words, $\bar{Q}$ is perfect. The rest of the proof is based on the following

LEMMA 3. *Let I_1 be any closed subinterval of I_0 given above. For $\varepsilon > 0$, $x \in I_1$ and a positive integer n, let*

$$B(x, \varepsilon) = \{y \in I_1\colon |f(y)-f(x)| < \varepsilon|y-x|\}$$

and

$$H_n(\varepsilon) = \left\{x \in I_1\colon |B(x, \varepsilon)\cap J| > \frac{2}{3}|J| \text{ for every } J \subset I_1 \text{ such that } x \in J \text{ and } |J| < \frac{1}{n}\right\},$$

where $|\cdot|$ means the Lebesgue measure of the set inside the bars. Then we have

(a) *If x, y are in $\overline{H_n(\varepsilon)}\cap K$ with $|x-y| < 1/n$, then*

$$|f(y)-f(x)| \leqslant 3\varepsilon|y-x|.$$

(b) *If $x \in \overline{H_n(\varepsilon)}\cap K$, then*

$$|B(x, 3\varepsilon)\cap J| > \tfrac{1}{2}|J|$$

for every $J \subset I_1$ with $x \in J$ and $|J| < 1/n$.

This lemma plays the same role as the lemma that O'Malley stated in [3], but it is different from the latter. The proof of the lemma above should be easier than O'Malley's. However, to make it clear, we sketch the proof of Lemma 3 as follows:

The conclusion (a) follows from (b) easily. It suffices to prove (b). Let ε and n be fixed and $z \in \overline{H_n(\varepsilon)}\cap K$.

Case (i). There exists a strictly decreasing sequence $\{x_k\}$ in $H_n(\varepsilon)$ with limit z and $0 < x_1 - z < 1/n$. We break the proof into five steps.

Step I. *$B(x_k, \varepsilon)$ has a positive upper density at z for each k.*

Proof. For fixed k and each positive integer j, the interval $[x_{k+j}, x_k]$ has length less than $1/n$ and contains the points x_{k+j} and x_k which are in the set $H_n(\varepsilon)$. It can be shown that

$$|f(x_{k+j})-f(x_k)| < \varepsilon(x_k - x_{k+j}). \tag{1}$$

With the aid of (1), we can show that

$$B(x_k, \varepsilon)\cap[z, x_{k+j}] \supset B(x_{k+j}, \varepsilon)\cap[z, x_{k+j}].$$

Thus

$$|B(x_k, \varepsilon)\cap[z, x_{k+j}]| > \frac{2}{3}(x_{k+j}-z).$$

Since $x_{k+j} \to z$ as $j \to \infty$, $B(x_k, \varepsilon)$ has an upper density at least 2/3 at z.

Step II. $|f(z)-f(x_k)| \leqslant \varepsilon(x_k - z)$ *for each* k.

Proof. $z \in K$ implies that f is approximately continuous at z. There exists a measurable set A such that A has density 1 at z and

$$\lim_{x \to z, x \in A} f(x) = f(z).$$

For fixed k, $A \cap B(x_k, \varepsilon)$ has a positive upper density at z and

$$|f(y)-f(x_k)| < \varepsilon |y - x_k| \quad \text{for every } y \in A \cap B(x_k, \varepsilon).$$

Keeping $y \in A \cap B(x_k, \varepsilon)$ and letting $y \to z$, we obtain

$$|f(z)-f(x_k)| \leqslant \varepsilon(x_k - z). \tag{2}$$

Step III. *For every* $\delta \in (0, 1/n)$ *such that* $z + \delta \in I_1$ *we have*

$$|B(z, 3\varepsilon) \cap [z, z+\delta]| > \tfrac{1}{2}\delta. \tag{3}$$

Proof. Let such δ be given; then there exists a k_0 such that $x_k < z+\delta$ for all $k \geqslant k_0$. If $y \in B(x_k, \varepsilon)$ and $y > x_k$, then

$$|f(y)-f(x_k)| < \varepsilon(y - x_k).$$

We see from this and (2) that

$$|f(y)-f(z)| < \varepsilon(y-z) \quad \text{for } y \in B(x_k, \varepsilon) \cap [x_k, z+\delta]$$

whenever $k \geqslant k_0$. Hence for $k \geqslant k_0$

$$B(z, \varepsilon) \cap [z, z+\delta] \supset B(x_k, \varepsilon) \cap [x_k, z+\delta]$$

and

$$|B(z, \varepsilon) \cap [z, z+\delta]| > \frac{2}{3}(z+\delta - x_k).$$

Let $k \to \infty$; then

$$|B(z, \varepsilon) \cap [z, z+\delta]| \geqslant \frac{2}{3}\delta > \frac{1}{2}\delta.$$

Since $B(z, 3\varepsilon) \supset B(z, \varepsilon)$, we have (3).

Step IV. *For every* $\delta \in (0, 1/n)$ *such that* $z - \delta \in I_1$ *we have*

$$|B(z, 3\varepsilon) \cap [z-\delta, z]| > \tfrac{1}{2}\delta.$$

Remark. The first three steps are parallel to the proof of a lemma in [2] (p. 85-86). The statement of Step IV is analogous to that of Step III, but the proof is not similar at all.

Proof of Step IV. Let $\delta \in (0, 1/n)$ with $z - \delta \in I_1$ be given; then there exists a k_1 such that $0 < x_k - z < \min\{1/n - \delta, \delta/8\}$ for all $k \geqslant k_1$.

Fix any $k \geqslant k_1$; $[z-\delta, x_k]$ is a closed interval in I_1 containing x_k with length less than $1/n$, and hence

$$|B(x_k, \varepsilon) \cap [z-\delta, x_k]| > \frac{2}{3}(x_k - z + \delta).$$

Noting that $z-\delta < z-\delta/8 < z < x_k < z+\delta/8$, we see that

$$x_k - \left(z - \frac{\delta}{8}\right) < \left(z + \frac{\delta}{8}\right) - \left(z - \frac{\delta}{8}\right) = \frac{1}{4}\delta = \frac{1}{4}(z-(z-\delta))$$
$$< \frac{1}{4}(x_k - z + \delta) < \frac{2}{3}(x_k - z + \delta).$$

Hence there exists a $y \in B(x_k, \varepsilon) \cap [z-\delta, z-\delta/8)$. That is, there exists a $y \in [z-\delta, z-\delta/8)$ with

$$|f(y) - f(x_k)| < \varepsilon(x_k - y) = \varepsilon(x_k - z) + \varepsilon(z - y).$$

In view of (2), we obtain

$$|f(y) - f(z)| < 2\varepsilon(x_k - z) + \varepsilon(z-y) < 2\varepsilon\frac{\delta}{8} + \varepsilon(z-y) < 3\varepsilon(z-y),$$

that is, $y \in B(z, 3\varepsilon)$. More precisely,

$$B(z, 3\varepsilon) \cap [z-\delta, z] \supset B(x_k, \varepsilon) \cap \left[z-\delta, z-\frac{1}{8}\delta\right)$$
$$= B(x_k, \varepsilon) \cap [z-\delta, x_k] - B(x_k, \varepsilon) \cap \left[z - \frac{1}{8}\delta, x_k\right].$$

Thus

$$|B(z, 3\varepsilon) \cap [z-\delta, z]| \geqslant |B(x_k, \varepsilon) \cap [z-\delta, x_k]| - \left(x_k - z + \frac{1}{8}\delta\right)$$
$$> \frac{2}{3}(x_k - z + \delta) - \left(x_k - z + \frac{1}{8}\delta\right) = \left(\frac{2}{3} - \frac{1}{8}\right)\delta - \frac{1}{3}(x_k - z)$$
$$> \left(\frac{2}{3} - \frac{1}{8}\right)\delta - \frac{1}{3}\cdot\frac{1}{8}\delta = \frac{1}{2}\delta.$$

Step V. ***The conclusion*** (b) ***holds.***

This is immediate from Steps III and IV.

Case (ii). There is no strictly decreasing sequence in $H_n(\varepsilon)$ with limit z. If $z \in H_n(\varepsilon)$, then the conclusion (b) is obvious. If $z \notin H_n(\varepsilon)$, then there must be a strictly increasing sequence in $H_n(\varepsilon)$ with limit z. The proof is similar to the above one. Thus the lemma is proved.

Now we return to the proof of our theorem. We are dealing with the case where $\bar{Q}$ is a perfect set. Using Lemma 3 and recalling that $\bar{Q} \subset K$, we check easily that the proof differs very little from that given by O'Malley in [3]. Therefore, we choose to omit the details.

COROLLARY. *Let f, defined on $[a, b]$, satisfy the conditions of our theorem. Then there exists an $x_0 \in (a, b)$ such that f is differentiable there and $f(b) - f(a) = f'(x_0)(b-a)$.*

Proof. Under our conditions, Preiss proved a mean value theorem ([4], Theorem 6). This corollary follows readily from this mean value theorem and our theorem above. It is a strong form of the former.

REFERENCES

[1] N. C. Manna and S. N. Mukhopadhyay, *On approximate derivatives and their properties*, Acta Mathematica Academiae Scientiarum Hungaricae 22 (1971), p. 309-320.

[2] R. J. O'Malley, *A density property with applications*, Transactions of the American Mathematical Society 199 (1974), p. 75-87.

[3] — *The set where an approximate derivative is a derivative*, Proceedings of the American Mathematical Society 54 (1976), p. 122-124.

[4] D. Preiss, *Approximate derivatives and Baire classes*, Czechoslovak Mathematical Journal 21 (1971), p. 373-382.

[5] S. Saks, *Theory of the integral*, Warszawa - Lwów 1937.

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