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A REMARK ON Λ-SYSTEMS IN BANACH SPACES

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1. Introduction. In [1] Davis, Dean, and Singer studied non-complemented subspaces of Banach spaces and showed that if a Banach space X has a Λ-system (see section 2 for the definition) then X has a non-complemented subspace. In the same paper [1] (p. 307) the authors ask whether some subsequence of every Λ-system is basic (i.e. is a Schauder basis for its closed linear span). The purpose of this note is to answer this question negatively by showing that if $1 < p \neq 2 < +\infty$, then the space l^p has a Λ-system no subsequence of which is basic.

2. Definitions and preliminary results. The *projection constant* $\lambda(X)$ for a subspace X of a Banach space Y is defined as the infimum of the norms of all projections from Y onto X [1]. If X is not complemented in Y, $\lambda(X) = +\infty$.

Denote by $[x_i]$ the closed linear span of the sequence (x_i) in X.

A linearly independent sequence (x_i) in a Banach space X is called a *Λ-system* [1] if $\lambda([x_1, \ldots, x_n]) \xrightarrow{n} \infty$ and $[x_i] = X$.

Davis, Dean, and Singer have shown that if $1 \leqslant p \neq 2 < +\infty$, then l^p (the space of all real p-th power summable sequences) has a Λ-system [1]. Hence any Banach space linearly homeomorphic to l^p also has such a sequence.

3. The construction. Let e_1 denote the unit vector $(1, 0, 0, \ldots)$ in l^p. Then $l^p = [e_1] + [e_i]_{i=2}^{\infty}$ where $X = [e_i]_{i=2}^{\infty}$ is isometrically isomorphic to l^p. As we have mentioned, X has a Λ-system (x_i) for which we may assume $\|x_i\| = 1$ for all i. We claim the sequence $e_1, x_1, x_2, \ldots$ is a Λ-system in l^p.

Clearly this sequence is linearly independent and its closed linear span is l^p. We need only show $\lambda([e_1, x_1, x_2, \ldots, x_n]) \xrightarrow{n} +\infty$. To do this, let P_n: $l^p \to [e_1, x_1, x_2, \ldots, x_n]$ be a projection and define Q_n: $X \to [e_1, x_1, x_2, \ldots, x_n]$ to be the restriction of P_n to X. Define

$$R_n\colon\ [e_1, x_1, \ldots, x_n] \to [x_1, \ldots, x_n]$$

by

$$R_n(a_0 e_1 + a_1 x_1 + \ldots + a_n x_n) = a_1 x_1 + \ldots + a_n x_n.$$

Now for any n,

$$\|P_n\| \geqslant \|Q_n\| = \sup_{\substack{\|x\|\leqslant 1 \\ x\in X}} \|Q_n(x)\|.$$

But $Q_n(x) = a_0 e_1 + a_1 x_1 + \ldots + a_n x_n$ where $a_0 = \langle Q_n x, e_1\rangle$ since $\langle e_1, x_i\rangle = 0$ for all $i = 1, 2, \ldots, n$ by definition of X. Hence

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| = \|Q_n x - a_0 e_1\| \leqslant \|Q_n x\| + |a_0| \leqslant 2\|Q_n x\|$$

(since $|a_0| = |\langle Q_n x, e_1\rangle| \leqslant \|Q_n x\|$), and so

$$\|Q_n(x)\| \geqslant \tfrac{1}{2}\Big\|\sum_{i=1}^{n} a_i x_i\Big\| = \tfrac{1}{2}\|R_n Q_n x\|.$$

It follows that $\|P_n\| \geqslant \frac{1}{2}\|R_n Q_n\|$, and since $R_n Q_n$ is a projection from X onto $[x_1, x_2, \ldots, x_n]$ we have $\|P_n\| \geqslant \frac{1}{2}\lambda([x_1, x_2, \ldots, x_n]) \xrightarrow{n} \infty$ and so $e_1, x_1, x_2, \ldots$ is a Λ-system since P_n was arbitrary.

Now let $z_1 = e_1$ and $z_{i+1} = x_i$ for $i = 1, 2, \ldots$ Then $\langle z_i, e_1\rangle \geqslant 0$ for all $i = 1, 2, \ldots$ Let $q_n = z_n/2^{n-1}$ for $n = 1, 2, \ldots$ Clearly the set (q_i) is linearly independent, $[q_i] = l^p$, and $\lambda([q_1, \ldots, q_n]) \xrightarrow{n} +\infty$.

It follows, therefore, that the set (w_n) in l^p defined by $w_n = \sum_{i=1}^{n} q_i$, $n = 1, 2, \ldots$, is also linearly independent, $[w_n] = l^p$, and $\lambda([w_1, \ldots, w_n]) \xrightarrow{n} +\infty$; i.e. (w_n) is a Λ-system in l^p for which $\sup_n \|w_n\| \leqslant 2$.

If $1 < p \neq 2 < +\infty$, we claim no subsequence of the Λ-system (w_n) is a basic sequence. For, if (w_{n_k}) is basic then it must converge weakly to zero in l^p since l^p is reflexive [2], [3]. But for all k,

$$\langle w_{n_k}, e_1\rangle = \Big\langle \sum_{i=1}^{n_k} q_i, e_1\Big\rangle = \sum_{i=1}^{n_k} \langle q_i, e_1\rangle \geqslant 1$$

by construction of (q_i), and so (w_{n_k}) does not converge weakly to zero in l^p.

REFERENCES

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