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THE ONLY CONTINUOUS VOLTERRA RIGHT INVERSES IN $C_c[0,1]$ OF THE OPERATOR $\frac{d}{dt}$ ARE $\int_a^t$

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Let $C_c[0,1]$ be the Banach space of all complex-valued continuous functions defined on the closed interval $[0,1]$ with the usual norm sup. Operators of the form

$$(Rx)(t) = \int_a^t x(u)\,du \tag{1}$$

are right inverses for the operator $D = \frac{d}{dt}$, however not all. For instance, if $\alpha_1, \ldots, \alpha_n$ are complex numbers such that $\alpha_1 + \ldots + \alpha_n = 1$ and $R_1, \ldots, R_n$ are arbitrary right inverses for D then the operator $\alpha_1 R_1 + \ldots + \alpha_n R_n$ is again a right inverse for D.

We say that an operator A is a *Volterra operator* if for every scalar λ the operator $I - \lambda A$ is invertible.

The following question has been posed in [5] by the first of the present authors:

Are all Volterra right inverses of the operator $D = \frac{d}{dt}$ in the space $C_c[0,1]$ of the form $R = \int_a^t$ $(0 \leqslant a \leqslant 1)$?

The present paper gives the positive answer to this question.

In general case the following characterization of all right inverses for a given right invertible operator D in a linear space is given in [3]:

Every right inverse of the operator D is of the form

$$R_1 = R + FA, \tag{2}$$

where F is an initial operator (i.e. projection onto $\ker D$) corresponding to a right inverse R of D and A is an arbitrary linear operator defined on the whole space.

Using this characterization we obtain Dimovski's characterization (cf. [1]) of all continuous right inverses for the operator $D = \frac{d}{dt}$ in $C_c[0, 1]$:

$$(Rx)(t) = \int_a^t x(S)\,ds + \psi(x), \tag{3}$$

where $0 \leqslant a \leqslant 1$, and ψ is an arbitrary continuous linear functional defined on the space $C_c[0, 1]$.

On the other hand, the Riesz characterization of continuous linear functionals leads to the following form of every right inverse for the operator $D = \frac{d}{dt}$:

$$(Rx)(t) = \int_a^t x(s)\,ds + \int_0^1 x(s)\,d\mu_s, \tag{4}$$

where μ is a complex-valued measure.

Consider now an arbitrary function $x_z(t) = e^{zt}$, where z plays a role of a complex parameter. We have for $z \neq 0$:

$$\begin{aligned}(I - zR)\,x_z(t) &= e^{zt} - z\left[\frac{1}{z}e^{zt} - \frac{1}{z}e^{za}\right] - z\int_0^1 e^{zt}\,d\mu_t \\ &= e^{za} - z\int_0^1 e^{zt}\,d\mu_t.\end{aligned}$$

Observe that the function

$$G(z) = e^{za} - z\int_0^1 e^{zt}\,d\mu_t \tag{5}$$

for a Volterra right inverse R does not vanish on the whole complex plane. Moreover, $G(z)$ is an entire function and

$$0 < |G(z)| \leqslant e^{2|z|} \quad \text{for } z \text{ large enough}, \tag{6}$$

i.e. $G(z)$ is a function of order 1. This and a classical theorem (cf. Markuševič [2], p. 514) together imply that $G(z)$ is of the form $Ce^{\alpha z}$, C, α are constants. Hence

$$z\int_0^1 e^{zt}\,d\mu_t = e^{za} - Ce^{z\alpha}. \tag{7}$$

Putting in (7) $z = 0$ we find that $C = 1$ and

$$\int_0^1 ze^{zt}\,d\mu_t = e^{za} - e^{z\alpha} = z\int_\alpha^a e^{zt}\,dt. \tag{8}$$

This and the density of the set $\operatorname{lin}\{e^{zt}\}_{z\in C}$ in $C_c[0, 1]$ together imply that the measure μ is uniquely determined and

$$\int_0^1 x(t)\,d\mu_t = \int_\alpha^a x(s)\,ds \quad \text{for } x\in C_c[0, 1]. \tag{9}$$

Hence

$$(Rx)(t) = \int_a^t x(s)\,ds + \int_\alpha^a x(s)\,ds = \int_\alpha^t x(s)\,ds \quad \text{for } x\in C_c[0, 1]. \tag{10}$$

Therefore we have

THEOREM 1. *Every continuous Volterra right inverse R for the operator $D = \frac{d}{dt}$ in the space $C_c[0, 1]$ is of the form* (1).

A continuous right inverse R of the operator $\frac{d}{dt}$ can have exactly one eigenvalue. For example the operator

$$Rx = \int_a^t x(s)\,ds + bx(a), \quad 0 \leqslant a \leqslant 1, \tag{11}$$

has for $b \neq 0$ a unique eigenvalue $1/b$ corresponding to the eigenvector $e^{t/b}$. Observe that if a right inverse R is not of the form (11), then the corresponding $G(z)$ is a transcendental function of order 1, which is not of the form $A+p(z)e^{P(z)}$, where $p(z)$ and $P(z)$ are polynomials. Thus by Picard Theorem (cf. [2], p. 515) the function $G(z)$ admits every complex value including 0 at infinite number of points. This implies

THEOREM 2. *If a continuous right inverse R of the operator $\frac{d}{dt}$ has two eigenvalues, then it has an infinite number of eigenvalues.*

Examples of non-Volterra right inverses can be found, for instance, in [4] and [5].

Note that there is a one-to-one correspondence between right inverses of the form $\int_a^t$ and initial operators of the form: $(Fx)(t) = x(a)$ for an $a\in[0, 1]$. Therefore Theorem 2 implies

COROLLARY 1. *If F is an initial operator for $D = \frac{d}{dt}$ (i.e. a projection onto* *the space of constant functions) such that the induced right inverse is not of the form* $\int_a^t$ *then the corresponding initial value problem*

$$\frac{d}{dt}x - \lambda x = 0 \qquad (\lambda \in \mathbf{C}),$$

$$Fx = x_0 \qquad (x_0 \textit{ is a given constant})$$

is ill-posed and has infinitely many complex eigenvalues, i.e. infinitely many eigenvectors.

In the space $C_r[0, 1]$ of all real-valued functions defined on the interval $[0, 1]$ and considered as a linear space over reals Theorem 1 does not hold, as follows from

Example 1. Let $0 \leqslant a < b \leqslant 1$ and let

$$(Rx)(t) = \int_a^t x(s)\,ds - \tfrac{1}{2}\int_a^b x(s)\,ds \qquad \text{for all } x \in C_r[0, 1].$$

We shall prove that R is a Volterra operator.

Since R is a sum of a Volterra operator and of a finite dimensional operator, we conclude that R is a Volterra operator if and only if R does not have eigenvectors (cf. [7]).

Evidently, R is a right inverse for the operator $\frac{d}{dt}$. This means that every eigenvector of R is an eigenvector of $\frac{d}{dt}$, hence is of the form $ce^{\alpha t}$. Thus it is enough to show that functions $e^{\alpha t}$ are not eigenvectors of R for all real α. We find for all real λ:

$$(I - \lambda R)e^{\alpha t} = e^{\alpha t} - \lambda\frac{1}{\alpha}(e^{\alpha t} - e^{\alpha a}) + \lambda\frac{1}{2\alpha}(e^{\alpha b} - e^{\alpha a})$$

$$= \left(1 - \frac{\lambda}{\alpha}\right)e^{\alpha t} + \frac{\lambda}{\alpha}e^{\alpha a} + \frac{\lambda}{2\alpha}e^{\alpha b} - \frac{\lambda}{2\alpha}e^{\alpha a}.$$

The first component is equal to zero if and only if $\lambda = \alpha$. If we put $\lambda = \alpha$ then we obtain

$$(I - dR)e^{\alpha t} = \tfrac{1}{2}(e^{\alpha a} + e^{\alpha b}) > 0 \qquad \text{for all real } \alpha.$$

Therefore $e^{\alpha t}$ is not an eigenvector of the operator $I - \alpha R$ for $-\infty < \alpha < +\infty$. This implies that R is a Volterra operator in $C_r[0, 1]$.

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