

*ON ω -APPROXIMATELY CONTINUOUS
DENJOY-STIELTJES INTEGRAL*

BY

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1. Introduction. In this paper we consider a Denjoy type of Stieltjes integral, the ADS-integral, defined descriptively by the method of Saks (cf. [12], p. 241), using the concept of approximate derivative with respect to a non-decreasing function ω . A similar integral has been considered by Ridder [11], but he assumes that the monotone function α associated with ω is continuous.

In a recent paper Kubota [8] has shown that the approximately continuous Denjoy integral defined by him is the least general among the approximately continuous integrals having the Cauchy and Harnack properties and including the Lebesgue integral (cf. [9], Theorem 4). Here we shall establish the analogous property of the ADS-integral (ω).

2. Preliminaries. Let ω be a finite non-decreasing function defined on the real line X . Let S denote the set of points of continuity of ω and $D = X \setminus S$. Then D is at most countable. Let S_0 denote the union of pairwise disjoint open intervals on each of which ω is constant, let S_0^- and S_0^+ denote the left and the right end points of all such intervals, respectively, and $S_1 = S_0^- \cup S_0^+$.

Let \mathcal{C} denote the family of open intervals (a, b) and the empty set \emptyset . Then the non-negative set function τ , defined on \mathcal{C} by $\tau(\emptyset) = 0$ and $\tau(a, b) = \omega(b-) - \omega(a+)$, determines the "method I outer measure" (see [10], p. 90) ω^* defined for $E \subset X$ by

$$\omega^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \tau(I_k) : \bigcup_{k=1}^{\infty} I_k \supset E, I_k \in \mathcal{C} \right\}.$$

It can be easily shown by using Theorem 13.8 of [10] that ω^* is a metric outer measure, so that Borel sets are measurable (ω^*) (see [10], Corollary 13.2.1, p. 104), and hence ω^* is regular (see [10], Corollary 12.3.1, p. 98).

Definition 2.1. Let $A \subset X$. For $x < y$, we write

$$d(x, y) = \begin{cases} \omega^*(A \cap (x, y]) / \omega^*(x, y] & \text{if } \omega^*(x, y] \neq 0, \\ 0 & \text{if } A \cap (x, y] = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\lim_{y \rightarrow x+} d(x, y)$, if it exists, is the *right ω^* -density* of A at x . The *left ω^* -density* of A at x is defined analogously. If the right and left ω^* -densities of A at x exist and are equal, the common value is the ω^* -density of A at x . If A has ω^* -density 1 or 0 at x , then x is called, accordingly, a *point of density* (ω^*) or a *point of dispersion* (ω^*) of A .

Two sets A and B are ω -separated (see [3], p. 347) if, for every $\varepsilon > 0$, there exist open sets $G_1 \supset A$ and $G_2 \supset B$ with $\omega^*(G_1 \cap G_2) < \varepsilon$.

Proceeding, with minor modifications, as in Theorems 2.20, 5.2, 5.4 of [6] (p. 59, 114 and 116, respectively) and using Lemma 2 of [5], we get

THEOREM 2.1. *If A and B are ω -separated and $A \cup B$ is measurable (ω^*), then A and B are also measurable (ω^*). If $A \subset S$, then A has ω^* -density unity a.e. (ω^*) on A . If $A \cap B \subset S$ and A has ω^* -density 0 a.e. (ω^*) on B , then A and B are ω -separated.*

Definition 2.2. Let f be an extended real-valued function defined on $A \subset X$ and let $\xi \in X$ be given. The g.l.b. of real numbers k for which the set

$$\{x: x \in A, x > \xi, f(x) > k\}$$

has ω^* -density 0 at ξ is denoted by $\bar{f}_\omega(\xi+)$ and called the *upper right ω -approximate limit* of f at ξ . The l.u.b. of real numbers k for which the set

$$\{x: x \in A, x > \xi, f(x) < k\}$$

has ω^* -density 0 at ξ is denoted by $\underline{f}_\omega(\xi+)$ and called the *lower right ω -approximate limit* of f at ξ . If $\bar{f}_\omega(\xi+) = \underline{f}_\omega(\xi+)$, the common value is denoted by $f_\omega(\xi+)$ and it is called the *right ω -approximate limit* of f at ξ . The symbols $\bar{f}_\omega(\xi-)$, $\underline{f}_\omega(\xi-)$ and $f_\omega(\xi-)$ have analogous meaning.

When $\xi \in A$, f is said to be *ω -approximately continuous* at ξ if

$$f_\omega(\xi-) = f_\omega(\xi+) = f(\xi) \neq \pm \infty$$

or

$$\frac{1}{2}[f_\omega(\xi-) + f_\omega(\xi+)] = f(\xi) \neq \pm \infty$$

according to as $\xi \in S$ or $\xi \in D$.

Using Theorem 2.1 and the theorem of Lusin (cf. [12], p. 72) as in the Lebesgue case, with obvious modifications, we get

THEOREM 2.2. *A finite function f defined on an ω^* -measurable set A is measurable (ω^*) if and only if f is ω -approximately continuous a.e. (ω^*) on $A \cap S$. And if $f_\omega(x-)$ and $f_\omega(x+)$ exist finitely for every $x \in [a, b]$, then, for every $\xi \in [a, b]$, there exists a closed set A_ξ such that the complement \tilde{A}_ξ has ω^* -density 0 at ξ , and $f_\omega(x+)$ and $f_\omega(x-)$ are, respectively, the ordinary right-hand and left-hand limits of f at x relative to $A_\xi \cap [a, b]$ for every $x \in A_\xi$.*

We denote by \mathcal{F}_ω the family of finite functions f defined on X such that $f_\omega(\xi-)$ and $f_\omega(\xi+)$ exist finitely at each point $\xi \in D$. If $f \in \mathcal{F}_\omega$, we denote by $\overset{\circ}{f}$ the function which coincides with f on S , and if $\xi \in D$, then

$$\overset{\circ}{f}(\xi) = \frac{1}{2}[f_\omega(\xi-) + f_\omega(\xi+)].$$

Definition 2.3. Let $f \in \mathcal{F}_\omega$ and $\xi \in X$ be given. Write

$$\psi(f, \xi; x) = \begin{cases} [f(x) - \overset{\circ}{f}(\xi)]/[\omega(x) - \overset{\circ}{\omega}(\xi)] & \text{if } \omega(x) \neq \overset{\circ}{\omega}(\xi), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\lim_{x \rightarrow \xi} \psi(f, \xi; x) = f'_\omega(\xi),$$

if it exists, is the ω -derivative of f at ξ . The *approximate upper* and *lower right* or *left* ω -derivatives of f at ξ are defined and denoted by

$$AD^+ f_\omega(\xi) = \overline{\psi}_\omega(f, \xi; \xi+) \quad \text{and} \quad AD_+ f_\omega(\xi) = \underline{\psi}_\omega(f, \xi; \xi+)$$

or

$$AD^- f_\omega(\xi) = \overline{\psi}_\omega(f, \xi; \xi-) \quad \text{and} \quad AD_- f_\omega(\xi) = \underline{\psi}_\omega(f, \xi; \xi-).$$

If

$$AD^+ f_\omega(\xi) = AD_+ f_\omega(\xi) = AD^- f_\omega(\xi) = AD_- f_\omega(\xi),$$

the common value is the *approximate* ω -derivative ($\text{ap}f'_\omega(\xi)$) of f at ξ .

It can be easily seen that, for $f \in \mathcal{F}_\omega$ and $\xi \in D$,

$$\text{ap}f'_\omega(\xi) = [f_\omega(\xi+) - f_\omega(\xi-)]/[\omega(\xi+) - \omega(\xi-)].$$

Definition 2.4. Let $f \in \mathcal{F}_\omega$, $f_+(x) = f_\omega(x+)$ and $f_-(x) = f_\omega(x-)$ if $x \in D$, and $f_+(x) = f_-(x) = f(x)$ if $x \in S$. For any interval I with the end points x and x' ($x < x'$), write

$$f(I) = \begin{cases} f_+(x') - f_-(x) & \text{if } I = [x, x'], \\ f_-(x') - f_+(x) & \text{if } I = (x, x'), \\ f_+(x') - f_+(x) & \text{if } I = (x, x'], \\ f_-(x') - f_-(x) & \text{if } I = [x, x'). \end{cases}$$

Then f is said to be AC- ω on a set $E \subset X$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any set of pairwise disjoint intervals $\{I_k\}$

with end points on E and with $\sum \omega(I_k) < \delta$, we have $\sum |f(I_k)| < \varepsilon$. The function f is $\overline{\text{ACG-}\omega}$ on E if E is the union of a countable number of closed sets on each of which f is $\overline{\text{AC-}\omega}$.

Note. Jeffery [5] denotes by \mathcal{U} the class of functions f , defined and continuous on $S \cap [a, b]$, such that the unilateral limits $f(\xi -)$ and $f(\xi +)$ exist for every $\xi \in D \cap [a, b]$, and $f(x) = f(a +)$ or $f(b -)$ according to as $x < a$ or $x > b$. We may, however, define

$$f(x) = \frac{1}{2}[f(x-) + f(x+)] \quad \text{for } x \in D$$

so that $\mathcal{U} \subset \mathcal{F}_\omega$. Our definition of ω -derivative is different from that of Jeffery [5]. However, for $f \in \mathcal{U}$, it is easy to see that if $f'_\omega(x)$ exists in one sense, then it exists in the other sense and they have the same value. If $f = f \in \mathcal{F}_\omega$ is $\overline{\text{AC-}\omega}$ on $[a, b]$ and $[a, \beta] \subset (a, b)$, then, clearly, f is $\text{AC-}\omega$ on $[a, \beta]$ in the sense of Jeffery (see [5], Definition 1), and hence (see [2], Theorem 3.1, and [5], Theorem 2) f'_ω is Lebesgue-Stieltjes integrable (ω) on $[a, \beta]$ and

$$(1) \quad f(\beta+) - f(a-) = (\text{LS}) \int_a^\beta f'_\omega d\omega.$$

THEOREM 2.3 (cf. [2], Theorem 3.2). *A function $F \in \mathcal{F}_\omega$, which is $\overline{\text{ACG-}\omega}$ on a set E , fulfills condition $(N\omega)$, i.e. for every $H \subset E$ with $\omega^*(H) = 0$, the Lebesgue measure $|F[H]|$ of $F[H]$ is zero.*

This theorem is a generalization of Theorem 6.1 in [12], p. 225. Clearly, it is sufficient to prove the theorem under the hypothesis that F is bounded and $\overline{\text{AC-}\omega}$ on E . Also, $\omega^*(H) = 0$ implies that $H \subset S$. Noting these, the theorem can be proved in a way analogous to the proof of Lemma 2.1 in [7].

THEOREM 2.4 (cf. [12], Theorem 7.1, p. 203, and [7], Lemma 2.2). *Let F be a finite function defined on $[a, b]$ and such that*

- (i) $\overline{F}_\omega(x-) \leq F(x) \leq \underline{F}_\omega(x+)$ for every $x \in [a, b]$,
- (ii) $F[E]$ does not contain any interval, where E denotes the set of points $\xi \in (a, b)$ such that $F(x) \leq F(\xi)$ in some right neighbourhood of ξ .

Then F is non-decreasing on $[a, b]$.

Proof. We first show that F is non-decreasing on each interval

$$[a, \beta] \subset (S_0 \cup S_1) \cap [a, b].$$

Choose $\varepsilon > 0$ arbitrarily. Then

$$F(a) - \varepsilon < y_0 < F(a) \quad \text{for some } y_0 \notin F[E].$$

Let

$$\xi = \sup \{x: x \in [a, \beta], F(x) > y_0\}.$$

Since $y_0 < F(a) \leq \underline{F}_\omega(a+)$, the set $\{x: x \in (a, \beta), F(x) \leq y_0\}$ has ω^* -density 0 at a . This implies, as $\omega^*(a, \beta) = 0$, that $F(x) > y_0$ in some right neighbourhood of a . Thus $a < \xi \leq \beta$. If we assume that $F(\xi) < y_0$, then, by hypothesis, $\bar{F}_\omega(\xi-) < y_0$, so that the set $\{x: x \in (a, \xi), F(x) \geq y_0\}$ has ω^* -density 0 at ξ . This implies, as $\omega^*(a, \xi) = 0$, that $F(x) < y_0$ in some left neighbourhood of ξ , which contradicts the definition of ξ . Thus we must have $F(\xi) \geq y_0$. Now, if we assume that $\xi < \beta$, then $F(x) \leq y_0$ for all $x \in (\xi, \beta)$. This implies $F(\xi) \leq \underline{F}_\omega(\xi+) \leq y_0$, so that $F(\xi) = y_0$, which contradicts that $y_0 \notin F[E]$. Thus we have $\xi = \beta$ and, consequently, $F(\beta) \geq y_0 > F(a) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $F(\beta) \geq F(a)$.

We now consider the general case. Suppose, if possible, that $F(d_0) < F(c_0)$ while $a \leq c_0 < d_0 \leq b$. Then there is $y_0 \notin F[E]$ such that $F(d_0) < y_0 < F(c_0)$. We write

$$A = \{x: x \in [c_0, d_0], F(x) > y_0\}, \quad B = \{x: x \in [c_0, d_0], F(x) \geq y_0\}.$$

Then $A \subset B$ and $c_0 \in A$, and, for every $x \in A$, we have $y_0 < F(x) \leq \underline{F}_\omega(x+)$, so that A has right ω^* -density 1 at x . Now, starting with c_0 , we can choose in $A \cap [c_0, d_0]$ a strictly increasing sequence $\{c_n\}$ such that

$$(2) \quad \omega^*(A \cap (c_{n-1}, c_n]) \geq \frac{1}{2} \omega^*(c_{n-1}, c_n], \quad n = 1, 2, \dots,$$

equality holding only if $\omega^*(c_{n-1}, c_n] = 0$. Let $c_n \rightarrow a$ ($a \leq d_0$). Then, as $c_n \in A \subset B$, we get from (2)

$$(3) \quad \omega^*(B \cap [c_n, a]) \geq \frac{1}{2} \omega^*[c_n, a], \quad n = 0, 1, 2, \dots,$$

equality holding only if $\omega^*[c_n, a] = 0$. Relation (3) implies that a is not a point of dispersion (ω^*) of $B \cap [c_0, a]$. So we conclude from hypothesis (i) that $y_0 \leq \bar{F}_\omega(a-) \leq F(a)$. If $a = d_0$, we have a contradiction. Now let $a < d_0$. If $\omega^*[c_0, a] = 0$, then, by the first part of the proof, we get $F(a) \geq F(c_0) > y_0$. Therefore, if $F(a) = y_0$, then (3) gives

$$\omega^*(B \cap [c_0, a]) > \frac{1}{2} \omega^*[c_0, a],$$

and hence, as $F(a) = y_0 \notin F[E]$, there is $a' \in A \cap (a, d_0)$ with

$$\omega^*(B \cap [c_0, a']) > \frac{1}{2} \omega^*[c_0, a').$$

Let $\alpha_0 = a$ or a' according to as $F(a)$ is greater or equal to y_0 . Then $\alpha_0 \in A$ and

$$\omega^*(B \cap [c_0, \alpha_0]) \geq \frac{1}{2} \omega^*[c_0, \alpha_0],$$

equality holding only if $\omega^*[c_0, \alpha] = 0$. The process can be repeated with α_0 , and carrying on the process the point d_0 must be reached in a countable number of steps (cf. [4], Section 79, p. 115), and thus we get $F(d_0) \geq y_0$. This final contradiction establishes the theorem.

3. The $\overline{\text{ADS}}$ -integral (ω) .

Definition 3.1 (cf. [5], Definition 7). Let the function f be finite a.e. (ω^*) on $[a, b]$ and suppose that there exists a function $F \in \mathcal{F}_\omega$ such that

- (i) F is ω -approximately continuous at each point of $[a, b]$,
- (ii) F is $\overline{\text{ACG}}\text{-}\omega$ on $[a, b]$,
- (iii) $\text{ap } F'_\omega(x) = f(x)$ a.e. (ω^*) on $[a, b]$.

Then f is said to be *integrable on $[a, b]$ in the ω -approximately continuous Denjoy-Stieltjes sense*; F is called an *indefinite $\overline{\text{ADS}}$ -integral (ω)* of f on $[a, b]$ and we write

$$\overline{\text{ADS}} \int_a^b f d\omega = F(b) - F(a).$$

Definition 3.1 is justified by the following theorem:

THEOREM 3.1. *If P and M are any two indefinite $\overline{\text{ADS}}$ -integrals (ω) of f on $[a, b]$, then $P(b) - P(a) = M(b) - M(a)$.*

Proof. Let $\varepsilon > 0$ be given. Write $F(x) = P(x) - M(x) + \varepsilon \dot{\omega}(x)$. Then, clearly, F is ω -approximately continuous on $[a, b]$. Now, let $\xi \in D \cap [a, b]$. Then

$$\text{ap } P'_\omega(\xi) = f(\xi) = \text{ap } M'_\omega(\xi),$$

and so

$$(4) \quad P_\omega(\xi+) - P_\omega(\xi-) = M_\omega(\xi+) - M_\omega(\xi-).$$

Also we have

$$(5) \quad \begin{aligned} F_\omega(\xi+) &= P_\omega(\xi+) - M_\omega(\xi+) + \varepsilon \omega(\xi+), \\ F_\omega(\xi-) &= P_\omega(\xi-) - M_\omega(\xi-) + \varepsilon \omega(\xi-). \end{aligned}$$

Using (4) and (5), we easily deduce that

$$F_\omega(\xi-) < F(\xi) < F_\omega(\xi+).$$

Thus F satisfies condition (i) of Theorem 2.4 on $[a, b]$. Again, $\text{ap } F'_\omega(x) = \varepsilon > 0$ a.e. (ω^*) on $[a, b]$. Consequently, the set

$$E_0 = \{x: x \in (a, b), \text{AD}^+ F_\omega(x) \leq 0\}$$

is of ω -measure zero. Obviously, E_0 includes the set E defined in Theorem 2.4. Therefore, $\omega^*(E) = 0$. But F is, clearly, $\overline{\text{ACG}}\text{-}\omega$ on $[a, b]$. So, by Theorem 2.3, $|F[E]| = 0$ and, hence, $F[E]$ does not contain any interval. Hence, by Theorem 2.4, we get

$$P(b) - M(b) + \varepsilon \dot{\omega}(b) \geq P(a) - M(a) + \varepsilon \dot{\omega}(a),$$

whence it follows that

$$(6) \quad P(b) - P(a) \geq M(b) - M(a).$$

Interchanging P and M in (6), we have

$$(7) \quad M(b) - M(a) \geq P(b) - P(a).$$

Combining (6) and (7), we get the theorem.

Definition 3.2. Given a function f defined on $[a, b]$, we write

$$\omega_f(x) = f(x)[\omega(x+) - \omega(x-)] \quad \text{on } D \cap [a, b] \text{ and } 0 \text{ elsewhere.}$$

As a direct consequence of Definition 3.1, we get

THEOREM 3.2. Let f and g be $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$.

(i) If $f(x) = h(x)$ a.e. (ω^*) on $[a, b]$, then h is $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$, and

$$\overline{\text{ADS}} \int_a^b f d\omega = \overline{\text{ADS}} \int_a^b h d\omega.$$

(ii) If λ and μ are finite constants, then $\lambda f + \mu g$ is $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$, and

$$\overline{\text{ADS}} \int_a^b (\lambda f + \mu g) d\omega = \lambda \cdot \overline{\text{ADS}} \int_a^b f d\omega + \mu \cdot \overline{\text{ADS}} \int_a^b g d\omega.$$

(iii) If $a < c < b$, then f is $\overline{\text{ADS}}$ -integrable (ω) on $[a, c]$ and $[c, b]$, and

$$\overline{\text{ADS}} \int_a^b f d\omega = \overline{\text{ADS}} \int_a^c f d\omega + \overline{\text{ADS}} \int_c^b f d\omega.$$

THEOREM 3.3. If $\overline{\text{ADS}} \int_a^c f d\omega$ and $\overline{\text{ADS}} \int_c^b f d\omega$ exist, then $\overline{\text{ADS}} \int_a^b f d\omega$

exists.

Proof. Let P and M be indefinite $\overline{\text{ADS}}$ -integrals (ω) of f on $[c, b]$ and $[a, c]$, respectively. Let F be defined by

$$F(x) = \begin{cases} P(x) - P(c) & \text{if } x > c, \\ M(x) - M(c) & \text{if } x \leq c. \end{cases}$$

Suppose that $c \in D$. Noting that $\text{ap}P'_\omega(c) = f(c) = \text{ap}M'_\omega(c)$, we get

$$(8) \quad P_\omega(c+) - P_\omega(c-) = M_\omega(c+) - M_\omega(c-).$$

Also we have

$$(9) \quad \begin{aligned} F_\omega(c+) &= P_\omega(c+) - P(c) = P_\omega(c+) - \frac{1}{2}[P_\omega(c+) + P_\omega(c-)], \\ F_\omega(c-) &= M_\omega(c-) - M(c) = M_\omega(c-) - \frac{1}{2}[M_\omega(c+) + M_\omega(c-)]. \end{aligned}$$

From (8) and (9) we easily deduce

$$(10) \quad \overset{\circ}{F}(c) = F(c) \quad \text{and} \quad \text{ap } F'_\omega(c) = f(c).$$

With relations (10) at hand it can be easily checked that F is an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$. Hence the proof of the theorem is completed.

Note. By Theorem 2.2, an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$ is necessarily measurable (ω^*) on $[a, b]$, but it can be shown, as in the Lebesgue case, that the approximate ω -derivatives of a function F , which is measurable (ω^*) on $[a, b]$, are also measurable (ω^*) on $[a, b]$. Thus an $\overline{\text{ADS}}$ -integrable (ω) function is necessarily measurable (ω^*).

THEOREM 3.4. *If f is Lebesgue-Stieltjes integrable (ω) on $[a, b]$, then f is $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$, and*

$$(\text{LS}) \int_a^b f d\omega = \overline{\text{ADS}} \int_a^b f d\omega + \frac{1}{2} \omega_f(a) + \frac{1}{2} \omega_f(b).$$

Proof. Let the function F be defined on X by

$$F(x) = \begin{cases} (\text{LS}) \int_a^x f d\omega & \text{if } a \leq x \leq b, \\ 0 & \text{if } x < a, \\ F(b) & \text{if } x > b. \end{cases}$$

Then $\overset{\circ}{F}$ is an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$ and

$$\begin{aligned} \overline{\text{ADS}} \int_a^b f d\omega &= \overset{\circ}{F}(b) - \overset{\circ}{F}(a) = \frac{1}{2} [F(b+) + F(b-) - F(a+) - F(a-)] \\ &= F(b) - \frac{1}{2} \omega_f(a) - \frac{1}{2} \omega_f(b). \end{aligned}$$

COROLLARY 3.4.1. *If F is an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$, then F is constant on each interval of $S_0 \cap [a, b]$.*

Theorem 3.4 shows that the definite $\overline{\text{ADS}}$ and LS-integrals (ω) on $[a, b]$ of a function f integrable (ω) in both senses are not always equal. This is due to the fact that LS-integral is not necessarily an additive function of closed intervals. However, the definition of LS-integral (ω) can be modified so as to be an additive function of closed intervals as follows:

Definition 3.3. A function f defined on $[a, b]$ is L_ω -integrable on $[a, b]$ if it is LS-integrable (ω) on $[a, b]$, and

$$L_\omega \int_a^b f d\omega = (LS) \int_a^b f d\omega - \frac{1}{2} \omega_f(a) - \frac{1}{2} \omega_f(b).$$

THEOREM 3.5. If f is $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$ and $f(x) \geq 0$ a.e. (ω^*) on $[a, b]$, then f is LS-integrable (ω) on $[a, b]$.

Proof. Let F be an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$. Then we have

$$\text{ap } F'_\omega(x) = f(x) \geq 0 \quad \text{a.e. } (\omega^*) \text{ on } [a, b].$$

Now, for arbitrary $\varepsilon > 0$, by considering the function $F(x) + \varepsilon \omega(x)$, it can be shown, by using Theorem 2.4 as in Theorem 3.1, that F is non-decreasing on $[a, b]$. Therefore, F'_ω is LS-integrable (ω) on $[a, b]$ (cf. [3], Theorem 6.3, p. 358). But

$$F'_\omega(x) = \text{ap } F'_\omega(x) = f(x) \quad \text{a.e. } (\omega^*) \text{ on } [a, b].$$

Therefore, f is LS-integrable (ω) on $[a, b]$, which completes the proof of the theorem.

THEOREM 3.6. Suppose that $\{f_n\}$ is a non-decreasing sequence of $\overline{\text{ADS}}$ -integrable (ω) functions on $[a, b]$, and that the sequence

$$\left\{ \overline{\text{ADS}} \int_a^b f_n d\omega + \frac{1}{2} \omega_{f_n}(a) + \frac{1}{2} \omega_{f_n}(b) \right\}$$

is bounded from above. Then the function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

is $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$, and

$$\overline{\text{ADS}} \int_a^b f d\omega = \lim_{n \rightarrow \infty} \overline{\text{ADS}} \int_a^b f_n d\omega.$$

Proof. Since $f_n(x) - f_1(x) \geq 0$ and, by Theorem 3.2 (ii),

$$\overline{\text{ADS}} \int_a^b (f_n - f_1) d\omega$$

exists, so, by Theorem 3.5,

$$(LS) \int_a^b (f_n - f_1) d\omega$$

exists for every n . Now, using Theorem 3.4, it follows that the integrals

$$(\text{LS}) \int_a^b (f_n - f_1) d\omega, \quad n = 1, 2, \dots,$$

constitute a sequence bounded from above. Hence, by Lebesgue's monotone convergence theorem,

$$(\text{LS}) \int_a^b (f - f_1) d\omega$$

exists, and

$$(\text{LS}) \int_a^b (f - f_1) d\omega = \lim_{n \rightarrow \infty} (\text{LS}) \int_a^b (f_n - f_1) d\omega.$$

This implies, by virtue of Theorem 3.4, that

$$\overline{\text{ADS}} \int_a^b f d\omega$$

exists, and

$$\begin{aligned} \overline{\text{ADS}} \int_a^b f d\omega + \frac{1}{2} \omega_f(a) + \frac{1}{2} \omega_f(b) \\ = \lim_{n \rightarrow \infty} \left\{ \overline{\text{ADS}} \int_a^b f_n d\omega + \frac{1}{2} \omega_{f_n}(a) + \frac{1}{2} \omega_{f_n}(b) \right\}, \end{aligned}$$

whence the theorem follows by noting that

$$\omega_{f_n}(a) \rightarrow \omega_f(a) \quad \text{and} \quad \omega_{f_n}(b) \rightarrow \omega_f(b).$$

THEOREM 3.7. *Let f be $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$. If*

$$P(x) = \overline{\text{ADS}} \int_a^x f d\omega \quad \text{and} \quad M(x) = \overline{\text{ADS}} \int_x^b f d\omega,$$

then we have

$$P_\omega(a+) = \frac{1}{2} \omega_f(a) \quad \text{and} \quad M_\omega(b-) = \frac{1}{2} \omega_f(b).$$

Proof. If F is an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$, then

$$P(x) = F(x) - F(a) \quad \text{and} \quad M(x) = F(b) - F(x).$$

If $a \in S$, we have, as F is ω -approximately continuous at a ,

$$P_\omega(a+) = F_\omega(a+) - F(a) = 0 = \frac{1}{2} \omega_f(a).$$

If $a \in D$, note that $\text{ap } F'_\omega(a) = f(a)$, whence

$$P_\omega(a+) = F_\omega(a+) - F(a) = F_\omega(a+) - \frac{1}{2}[F_\omega(a+) + F_\omega(a-)] = \frac{1}{2}\omega_f(a).$$

This completes the proof of the first part. The proof of the second part of this theorem is analogous.

THEOREM 3.8 (cf. [9], Theorem 3). *Let f be $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$. Then, for every non-empty closed set $E \subset [a, b]$, there exists an interval (α, β) with $E \cap (\alpha, \beta) \neq \emptyset$ such that*

- (i) *f is LS-integrable (ω) on $E \cap [\alpha, \beta]$;*
- (ii) *if $I_k = [a_k, b_k]$ denotes the closed interval contiguous to $E \cap [\alpha, \beta]$ relative to $[\alpha, \beta]$, then*

$$\sum_k \left| \overline{\text{ADS}} \int_{I_k} f d\omega \right| < +\infty;$$

- (iii) *if x is a limit point of end points of $\{I_k\}$, then there exists a set $A_x \supset E \cap [\alpha, \beta]$ such that A_x has ω^* -density 0 at x and*

$$\lim_{k \rightarrow \infty} O(\omega\text{-}\overline{\text{ADS}}, f, A_x \cap I_k) = 0,$$

where $O(\omega\text{-}\overline{\text{ADS}}, f, A_x \cap I_k)$ denotes the oscillation on $A_x \cap I_k$ of the indefinite $\overline{\text{ADS}}$ -integral (ω) of f on I_k .

Proof. Let F be an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, b]$. Then we can write

$$[a, b] = \bigcup_{k=1}^{\infty} E_k,$$

where each E_k is closed and F is AC- ω on each E_k . Since

$$E = \bigcup_{k=1}^{\infty} (E \cap E_k),$$

there exist, by Baire's category theorem, an interval (α, β) with $E \cap (\alpha, \beta) \neq \emptyset$ and a positive integer n such that $E \cap [\alpha, \beta] \subset E \cap E_n$. Therefore, F is AC- ω on $E \cap [\alpha, \beta]$.

Neglecting the trivial case where $E \cap [\alpha, \beta]$ is a singleton, we can assume that $\alpha, \beta \in E$. We now write

$$H(x) = \begin{cases} F_\omega(a_k+) + \frac{F_\omega(b_k-) - F_\omega(a_k+)}{\omega(b_k-) - \omega(a_k+)} \{\dot{\omega}(x) - \omega(a_k+)\}, & x \in (a_k, b_k) \notin S_0; \\ F_\omega(a_k+), & x \in (a_k, b_k) \subset S_0; \\ F(x), & x \in E \cap [\alpha, \beta]; \\ F_\omega(\beta+), & x > \beta; \\ F_\omega(\alpha-), & x < \alpha. \end{cases}$$

Observe that F is ω -approximately continuous on $[a, b]$ and by Corollary 3.4.1, F is constant on the intervals of $S_0 \cap [a, b]$. Now, let $\varepsilon > 0$ be given. Then, as F is $\overline{\text{AC}}\text{-}\omega$ on $E \cap [a, \beta]$, there is a $\delta > 0$ as required by Definition 2.4. Given $c \in E \cap [a, \beta]$, we can find $\xi < c$ and $\eta > c$ such that

$$\omega(c-) - \omega(\xi-) < \delta \quad \text{and} \quad \omega(\eta+) - \omega(c+) < \delta.$$

Then, if $x \in (a_k, b_k) \subset (\xi, c)$, we have

$$|H(x) - F_\omega(c-)| \leq |F_\omega(a_k+) - F_\omega(c-)| + |F_\omega(b_k-) - F_\omega(a_k+)| < 2\varepsilon,$$

and if $x \in (a_m, b_m) \subset (c, \eta)$, we have

$$|H(x) - F_\omega(c+)| \leq |F_\omega(a_m+) - F_\omega(c+)| + |F_\omega(b_m-) - F_\omega(a_m+)| < 2\varepsilon.$$

Hence H is ω -approximately continuous on X . Clearly, H is $\overline{\text{AC}}\text{-}\omega$ on each (a_k, b_k) , and hence, corresponding to each $\varepsilon/2^{k+1}$, there is a $\delta_k > 0$ as required in Definition 2.4. We can find a positive integer N such that

$$\sum_{k=N+1}^{\infty} \{\omega(b_k+) - \omega(a_k-)\} < \delta.$$

Then, putting $\delta_0 = \min\{\delta, \delta_1, \delta_2, \dots, \delta_N\}$, we see that, for every set of pairwise disjoint intervals $\{J_k\}$ with $\sum \omega(J_k) < \delta_0$, we have $\sum |H(J_k)| < 2\varepsilon$. This implies that H is $\overline{\text{AC}}\text{-}\omega$ on X , and hence H'_ω is LS-integrable (ω) on X . But

$$H'_\omega(x) = \text{ap } F'_\omega(x) = f(x) \quad \text{a.e. } (\omega^*) \text{ on } E \cap [a, \beta].$$

Therefore, f is LS-integrable (ω) on $E \cap [a, \beta]$, which proves (i).

Again, H being $\overline{\text{AC}}\text{-}\omega$ on X , it can be shown in a way analogous to the proof of Theorem 5 in [1], p. 62, that it is of bounded variation on X . Therefore, F is BV on $E \cap [a, \beta]$, and thus we get condition (ii),

$$\sum \left| \overline{\text{ADS}} \int_{a_k}^{b_k} f d\omega \right| = \sum |F(b_k) - F(a_k)| < +\infty.$$

Finally, assume c is a limit point of end points of $\{I_k\}$. By Theorem 2.2, there exists a closed set A such that A has ω^* -density 0 at c and, for every $x \in A$, $F_\omega(x-)$ and $F_\omega(x+)$ are, respectively, the ordinary unilateral limits of F at x relative to A . Since F is $\overline{\text{AC}}\text{-}\omega$ on $E \cap [a, \beta]$, we can assume further that $A \supset E \cap [a, \beta]$. Now, given $\varepsilon > 0$, corresponding to every $x \in A$, we can find $\delta_x > 0$ such that

$$(11) \quad |F(x_2) - F(x_1)| < \frac{\varepsilon}{2}$$

for all $x_1, x_2 \in A$ with $[x_1, x_2] \subset (x - \delta x, x) \cup (x, x + \delta x)$. By the Heine-Borel theorem, there is a finite collection of the intervals $(x - \delta x, x + \delta x)$, say $\{(x_i - \delta x_i, x_i + \delta x_i)\}_{i=1}^m$, covering $A \cap [a, \beta]$. Thus we can find a positive integer N such that, for every $k \geq N$ for which $A \cap I_k \neq \emptyset$, there exists an i with

$$I_k \subset (x_i - \delta x_i, x_i + \delta x_i).$$

It follows, by using (11), that, for all $k \geq N$,

$$O(\overline{\omega\text{-ADS}}, f, A \cap I_k) \leq \frac{\varepsilon}{2} < \varepsilon.$$

This completes the proof of the theorem.

THEOREM 3.9 (cf. [9], Theorem 1). *Let f be $\overline{\text{ADS-integrable}}(\omega)$ on $[a, x]$ for every $x \in (a, b)$, and let $F(x)$ denote the definite $\overline{\text{ADS-integral}}(\omega)$ of f on $[a, x]$. If $F_\omega(b-)$ exists finitely and if $\omega_f(b)$ is finite, then f is $\overline{\text{ADS-integrable}}(\omega)$ on $[a, b]$ and we have*

$$\overline{\text{ADS}} \int_a^b f d\omega = F_\omega(b-) + \frac{1}{2} \omega_f(b).$$

Proof. Let $a = \beta_0 < \beta_1 < \beta_2 < \dots$ be a sequence converging to b . Then f is $\overline{\text{ADS-integrable}}(\omega)$ on each of the intervals $I_k = [\beta_k, \beta_{k+1}]$. Let F_k denote the indefinite $\overline{\text{ADS-integral}}(\omega)$ of f on I_k satisfying $F_k(\beta_k) = 0$. Write

$$H(x) = \begin{cases} F_0(x) & \text{if } x \leq \beta_1, \\ F_n(x) + \sum_{k=0}^{n-1} F_k(\beta_{k+1}) & \text{if } x \in I_n, n \geq 1, \\ F_\omega(b-) + \frac{1}{2} \omega_f(b) & \text{if } x = b, \\ F_\omega(b-) + \omega_f(b) & \text{if } x > b. \end{cases}$$

Then it is easily seen that, for $a < x < b$,

$$H(x) = \overline{\text{ADS}} \int_a^x f d\omega = F(x),$$

and H is an indefinite $\overline{\text{ADS-integral}}(\omega)$ of f on $[a, b]$. Thus f is $\overline{\text{ADS-integrable}}(\omega)$ on $[a, b]$, and we have

$$\overline{\text{ADS}} \int_a^b f d\omega = H(b) - H(a) = F_\omega(b-) + \frac{1}{2} \omega_f(b).$$

THEOREM 3.10. *Let f be $\overline{\text{ADS}}$ -integrable (ω) on $[x, b]$ for every $x \in (a, b)$, and let $F(x)$ be the definite $\overline{\text{ADS}}$ -integral (ω) of f on $[x, b]$. Suppose that $F_\omega(a+)$ exists finitely and $\omega_f(a)$ is also finite. Then f is $\overline{\text{ADS}}$ -integrable (ω) on $[a, b]$, and we have*

$$\overline{\text{ADS}} \int_a^b f d\omega = F_\omega(a+) + \frac{1}{2} \omega_f(a).$$

THEOREM 3.11 (cf. [9], Theorem 2, and [12], Theorem 5.1, p. 257). *Let f be LS-integrable (ω) on a closed set $Q \subset I = [a, b]$ and $\overline{\text{ADS}}$ -integrable (ω) on each of the closed intervals contiguous to Q relative to I . Let $a < \alpha$ and $\beta > b$, and let f be extended over $[\alpha, \beta]$ by defining $f(x) = 0$ for $x \in [\alpha, \beta] \setminus I$. Let $J_k = [\alpha_k, \beta_k]$ denote the interval contiguous to Q relative to $[\alpha, \beta]$, $A = \{\alpha_k\}$ and $B = \{\beta_k\}$. Suppose further that the following conditions are satisfied:*

(i) $\sum_k |\overline{\text{ADS}} \int_{J_k} f d\omega| < +\infty.$

(ii) *If x is a limit point of $A \cup B$, then there exists a set E_x containing all points of $A \cup B$ in some neighbourhood of x such that E_x has ω^* -density 0 at x and*

(*) $\lim_{k \rightarrow \infty} O(\omega\text{-}\overline{\text{ADS}}, f, E_x \cap J_k) = 0.$

Then f is $\overline{\text{ADS}}$ -integrable (ω) on I and we have

$$\begin{aligned} & \overline{\text{ADS}} \int_a^b f d\omega \\ &= (\text{LS}) \int_Q f d\omega + \sum_k \overline{\text{ADS}} \int_{J_k \cap I} f d\omega - \frac{1}{2} \sum_{\alpha_k \in I} \omega_f(\alpha_k) - \frac{1}{2} \sum_{\beta_k \in I} \omega_f(\beta_k). \end{aligned}$$

Proof. Since f is LS-integrable (ω) on Q and $\overline{\text{ADS}}$ -integrable (ω) on each J_k , we assert that f is finite a.e. (ω^*) on $[\alpha, \beta]$ and we must have

$$\sum_k |\omega_f(\alpha_k)| < +\infty \quad \text{and} \quad \sum_k |\omega_f(\beta_k)| < +\infty.$$

Put $I_x = [a, x]$ for $a < x \leq \beta$ and define H on X by

$$H(x) = \begin{cases} \sum_k \overline{\text{ADS}} \int_{J_k \cap I_x} f d\omega + \frac{1}{2} \sum_{\alpha_k \in (a, x)} \omega_f(\alpha_k) + \frac{1}{2} \sum_{\beta_k \in (a, x)} \omega_f(\beta_k) & \text{for } a < x \leq \beta, \\ 0 & \text{for } x \leq a, \\ H(\beta) & \text{for } x > \beta, \end{cases}$$

where $\{a_k\}$ and $\{b_k\}$ are enumerations of $A \setminus B$ and $B \setminus A$, respectively, and $\overline{\text{ADS}} \int_E f d\omega = 0$ if E is either the empty set or a singleton.

We shall show in the following steps that H is an indefinite $\overline{\text{ADS}}$ -integral (ω) of some function on $[\alpha, \beta]$.

Step I. To show that H is ω -approximately continuous on $[\alpha, \beta]$, we take an arbitrary but fixed point $c \in [\alpha, \beta]$.

I (a). Let $a_n \leq c < \beta_n$ for some n . Then, for all $x \in (c, \beta_n)$,

$$H(x) = H(c) + \overline{\text{ADS}} \int_c^x f d\omega.$$

This implies, by Theorem 3.7, that

$$H_\omega(c+) = H(c) + \frac{1}{2} \omega_f(c).$$

I (b). Let $c \in B \setminus A$ and suppose that c is a limit point of $A \cup B$. Then, given $\varepsilon > 0$, we can find, by hypotheses, a positive integer N and a $\delta > 0$ such that $(c, c + \delta)$ does not contain any end points of J_k for $k \leq N$ and

$$(12) \quad \begin{aligned} \sum_{k > N} \left| \overline{\text{ADS}} \int_{J_k} f d\omega \right| &< \varepsilon, & \sum_{a_k \in (c, c + \delta)} |\omega_f(a_k)| &< \varepsilon, \\ \sum_{b_k \in (c, c + \delta)} |\omega_f(b_k)| &< \varepsilon, & \text{and} & \quad O(\omega\text{-}\overline{\text{ADS}}, f, E_c \cap J_k) < \varepsilon \\ & & & \text{for } k > N, \end{aligned}$$

where E_c has ω^* -density 0 at c and E_c contains all end points of $J_k \subset (c, c + \delta)$. If $x \in \{E_c \cap (c, c + \delta)\} \setminus \bigcup_k J_k$, we have

$$\begin{aligned} H(x) - H(c) &= \sum_{k > N} \overline{\text{ADS}} \int_{J_k \cap (c, x)} f d\omega + \frac{1}{2} \sum_{a_k \in (c, x)} \omega_f(a_k) + \frac{1}{2} \sum_{b_k \in (c, x)} \omega_f(b_k) + \frac{1}{2} \omega_f(c), \end{aligned}$$

while, if $x \in E_c \cap J_n \cap (c, c + \delta)$ for some n , then $n > N$ and

$$\begin{aligned} H(x) - H(c) &= \sum_{J_k \subset (c, a_n]} \overline{\text{ADS}} \int_{J_k} f d\omega + \overline{\text{ADS}} \int_{a_n}^x f d\omega + \\ &\quad + \frac{1}{2} \sum_{a_k \in (c, x)} \omega_f(a_k) + \frac{1}{2} \sum_{b_k \in (c, x)} \omega_f(b_k) + \frac{1}{2} \omega_f(c). \end{aligned}$$

These relations together with (12) imply that

$$|H(x) - H(c) - \frac{1}{2} \omega_f(c)| < 3\varepsilon \quad \text{for all } x \in E_c \cap (c, c + \delta).$$

Thus

$$H_{\omega}(c+) = H(c) + \frac{1}{2}\omega_f(c),$$

which is also trivially true if c is an isolated point of $A \cup B$.

I (c). Let $c \in Q \setminus (A \cup B)$. Then, proceeding as in I(b), it can be seen that

$$H_{\omega}(c+) = H(c).$$

It can be shown analogously that in cases I(a) and I(b) we have $H_{\omega}(c-) = H(c) - \frac{1}{2}\omega_f(c)$, while in case I(c) we have $H_{\omega}(c-) = H(c)$. Thus we have

$$(13) \quad H_{\omega}(x \pm) = \begin{cases} H(x) \pm \frac{1}{2}\omega_f(x) & \text{if } x \in \bigcup_k J_k, \\ H(x) & \text{if } x \in Q \setminus (A \cup B). \end{cases}$$

Relations (13) imply that H is ω -approximately continuous at each point of $[\alpha, \beta]$. This completes the step I.

Step II. To show that H is $\overline{\text{ACG}}\text{-}\omega$ on $[\alpha, \beta]$, we note that

$$(14) \quad H(x) = H(a_k) + \overline{\text{ADS}} \int_{a_k}^x f d\omega \quad \text{for } x \in J_k, \quad k = 1, 2, \dots$$

So H is $\overline{\text{ACG}}\text{-}\omega$ on $\bigcup_k J_k$. We now define the function g on $[\alpha, \beta]$ by

$$g(x) = \begin{cases} [\omega(\beta_k-) - \omega(a_k+)]^{-1} \overline{\text{ADS}} \int_{J_k} f d\omega & \text{if } x \in (a_k, \beta_k) \notin S_0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then g vanishes on Q and is constant on each (a_k, β_k) . So, by (i), g is LS-integrable (ω) on $[\alpha, \beta]$, whence, by Theorem 3.4, g is $\overline{\text{ADS}}$ -integrable (ω) on $[\alpha, \beta]$. We define the function G on X by

$$G(x) = \begin{cases} \overline{\text{ADS}} \int_{\alpha}^x g d\omega & \text{if } \alpha < x \leq \beta, \\ G(\beta) & \text{if } x > \beta, \\ 0 & \text{if } x \leq \alpha. \end{cases}$$

Then, for $x \in Q \setminus (A \cup B)$, we have, by Theorem 3.4,

$$G(x) = (\text{LS}) \int_{\alpha}^x g d\omega = \sum_k \overline{\text{ADS}} \int_{J_k \cap I_x} f d\omega.$$

Again, let us define the function u on $[a, \beta]$ by

$$u(x) = \begin{cases} \frac{1}{2}f(x) & \text{if } x \in (A \setminus B) \cup (B \setminus A), \\ 0 & \text{elsewhere.} \end{cases}$$

Then, for $x \in Q \setminus (A \cup B)$, we have

$$\frac{1}{2} \sum_{a_k \in (a, x)} \omega_f(a_k) + \frac{1}{2} \sum_{b_k \in (a, x)} \omega_f(b_k) = (\text{LS}) \int_a^x u d\omega.$$

It follows now that, for all $x \in Q \setminus (A \cup B)$,

$$(15) \quad H(x) = (\text{LS}) \int_a^x g d\omega + (\text{LS}) \int_a^x u d\omega.$$

Therefore, H is $\overline{\text{AC}}\text{-}\omega$ on $Q \setminus (A \cup B)$. This completes the step II.

Step III. From (13) it follows that

$$(16) \quad \text{ap } H'_\omega(x) = \begin{cases} f(x) & \text{if } x \in D \cap (\bigcup_k J_k), \\ 0 & \text{if } x \in (D \cap Q) \setminus (A \cup B). \end{cases}$$

Again, the set $Q_1 = Q \setminus (A \cup B \cup D)$, being measurable (ω^*) , has ω^* -density one a.e. (ω^*) on itself. So it follows from (15) that

$$(17) \quad \text{ap } H'_\omega(x) = g(x) + u(x) = 0 \quad \text{a.e. } (\omega^*) \text{ on } Q_1.$$

Also, it follows from (14) that

$$(18) \quad \text{ap } H'_\omega(x) = f(x) \quad \text{a.e. } (\omega^*) \text{ on } (\bigcup_k J_k) \setminus (A \cup B).$$

Thus we have proved the desired properties of H .

Next, we define the function h on $[a, \beta]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in Q \setminus (A \cup B), \\ 0 & \text{if } x \in \bigcup_k J_k. \end{cases}$$

Then h is, clearly, LS-integrable (ω) on $[a, \beta]$ and, hence, $\overline{\text{ADS}}$ -integrable (ω) on $[a, \beta]$. Let M be the indefinite $\overline{\text{ADS}}$ -integral (ω) of h on $[a, \beta]$ satisfying $M(a) = 0$. Finally, let F be defined on X by

$$F(x) = H(x) + M(x).$$

Then we see that F is ω -approximately continuous and $\overline{\text{ACG}}\text{-}\omega$ on $[a, \beta]$. Also, as $\text{ap } M'_\omega(x) = h(x)$ a.e. (ω^*) on $[a, \beta]$, we have, by (16), (17) and (18),

$$\text{ap } F'_\omega(x) = f(x) \quad \text{a.e. } (\omega^*) \text{ on } [a, \beta].$$

Thus F is an indefinite $\overline{\text{ADS}}$ -integral (ω) of f on $[a, \beta]$. It follows, therefore, that f is $\overline{\text{ADS}}$ -integrable (ω) on $[a, \beta]$. Now let $\{c_k\}$ be an enumeration of $A \cap B$. Noting that $h(a) = h(\beta) = 0$ and each of the intervals $[a, a]$ and $[b, \beta]$ is contained in some J_k , we get

$$\begin{aligned} \overline{\text{ADS}} \int_a^\beta f d\omega &= F(\beta) - F(a) = H(\beta) + M(\beta) \\ &= (\text{LS}) \int_Q f d\omega + \sum_k \overline{\text{ADS}} \int_{J_k} f d\omega - \frac{1}{2} \sum_k \omega_f(a_k) - \frac{1}{2} \sum_k \omega_f(b_k) - \sum_k \omega_f(c_k). \end{aligned}$$

This is easily seen to reduce to

$$\overline{\text{ADS}} \int_a^b f d\omega = (\text{LS}) \int_Q f d\omega + \sum_k \overline{\text{ADS}} \int_{J_k \cap I} f d\omega - \frac{1}{2} \sum_{a_k \in I} \omega_f(a_k) - \frac{1}{2} \sum_{\beta_k \in I} \omega_f(\beta_k),$$

which is clearly independent of the choice of the interval $[a, \beta]$. This completes the proof of the theorem.

4. A characterization of the $\overline{\text{ADS}}$ -integral (ω).

Definition 4.1. Let \mathcal{F} denote the family of extended real-valued functions defined on X , and let \mathcal{G} denote the family of closed intervals on X . If T is a real-valued linear functional defined on a subset of $\mathcal{F} \times \mathcal{G}$, we write

$$\mathcal{F}(T, I) = \{f: (f, I) \in \text{domain of } T\},$$

and if $I = [a, \beta]$, we use notation $T(f, I)$ and $T_a^\beta(f)$ interchangeably. Let \mathcal{J}_ω denote the family of functionals T satisfying the following conditions:

- (i) If $f \in \mathcal{F}(T, I)$, then $f \in \mathcal{F}(T, J)$ for all $J \subset I$.
- (ii) If I_1 and I_2 are abutting intervals in \mathcal{G} and if $f \in \mathcal{F}(T, I_1) \cap \mathcal{F}(T, I_2)$, then

$$f \in \mathcal{F}(T, I_1 \cup I_2) \quad \text{and} \quad T(f, I_1 \cup I_2) = T(f, I_1) + T(f, I_2).$$

- (iii) If $f \in \mathcal{F}(T, I)$, then there exists a function F defined on X such that F is ω -approximately continuous at each point of I and $F(x) = T_a^x(f)$ for $x \in (a, \beta]$, where $I = [a, \beta]$.

Then, every $T \in \mathcal{J}_\omega$ is called an ω -approximately continuous integral, and the functions in $\mathcal{F}(T, I)$ are called T -integrable (ω) on I .

Note. The $\overline{\text{ADS}}$ -integral (ω) is certainly an ω -approximately continuous integral, and we denote this integral by T_0 . The LS-integral (ω) fails to be ω -approximately continuous due to condition (ii). However, the modified LS-integral (ω), namely the L_ω -integral, in the sense of Definition 3.3, is ω -approximately continuous.

Definition 4.2. If $T, T' \in \mathcal{J}_\omega$, we say that T includes T' on I , written $T \supset T'$ on I , if $f \in \mathcal{F}(T', I)$ implies $f \in \mathcal{F}(T, I)$ and $T(f, J) = T'(f, J)$ for every $J \subset I$, where $I, J \in \mathcal{G}$.

Definition 4.3 (cf. [9], Definition 2). The *Cauchy* (C ω) and the *Harnack* (H ω) properties of an integral $T \in \mathcal{J}_\omega$ are given by the following conditions:

(C ω) If $f \in \mathcal{F}(T, I)$ for every $I = [\gamma, \delta] \subset (a, \beta)$ and

$$\omega\text{-ap} \lim_{\gamma \rightarrow a+, \delta \rightarrow \beta-} T_\gamma^\delta(f)$$

exists finitely, and if $\omega_f(a)$ and $\omega_f(\beta)$ are finite, then $f \in \mathcal{F}(T, [a, b])$ and

$$T_a^\beta(f) = \omega\text{-ap} \lim_{\gamma \rightarrow a+, \delta \rightarrow \beta-} T_\gamma^\delta(f) + \frac{1}{2}\omega_f(a) + \frac{1}{2}\omega_f(\beta).$$

(H ω) Let f be LS-integrable (ω) on a closed set $Q \subset [a, \beta]$ and T -integrable (ω) on each of the closed intervals contiguous to Q relative to $[a, b]$. Let $J_k = [a_k, \beta_k]$ be the interval contiguous to Q relative to $[a, \beta]$, where $[a, b] \subset (a, \beta)$. Suppose also that the following conditions are satisfied:

(i) $\sum_k |T(f, J_k)| < +\infty.$

(ii) If x is a limit point of end points of $\{J_k\}$, then there exists a set E_x containing all end points of $\{J_k\}$ in some neighbourhood of x such that E_x has ω^* -density 0 at x and

$$\lim_{k \rightarrow \infty} O(T, f, E_x \cap J_k) = 0.$$

Then $f \in \mathcal{F}(T, [a, b])$ and we have

$$T_a^\beta(f) = (\text{LS}) \int_Q f d\omega + \sum_k T \int_{J_k \cap [a, b]} f d\omega - \frac{1}{2} \sum_{a_k \in [a, b]} \omega_f(a_k) - \frac{1}{2} \sum_{\beta_k \in [a, b]} \omega_f(\beta_k).$$

Note. Combining Theorems 3.9 and 3.10, we see that the T_0 -integral has property (C ω), and Theorem 3.11 shows that the T_0 -integral has property (H ω).

THEOREM 4.1 (cf. [9], Lemma 2). Let $T \in \mathcal{J}_\omega$ have property (C ω), and let $I \in \mathcal{G}$ be given. Suppose that, for every interior point x of I , we can find a closed interval J_x containing x in its interior such that $T \supset T_0$ on J_x . Then $T \supset T_0$ on I .

The proof is identical to that of Lemma 2 in [9].

THEOREM 4.2. Let $T \in \mathcal{J}_\omega$ have properties (C ω) and (H ω). Then $T \supset T_0$ on every $I \in \mathcal{G}$.

Proof. Let $J = [a, \beta]$. By Theorem 4.1, it suffices to show that $T \supset T_0$ on every $[a, b] \subset (a, \beta)$. Suppose, if possible, that $T \not\supset T_0$ on

$I = [a, b]$. Let Q be the set of points $x \in I$ such that $T \not\supset T_0$ on any interval containing x . Then $Q \neq \emptyset$, $Q \subset (a, \beta)$, and Q is clearly closed. Now, take an arbitrary but fixed $f \in \mathcal{F}(T_0, J)$. Then, by Theorem 3.8, there exists an interval (a_1, b_1) such that $Q \cap (a_1, b_1) \neq \emptyset$ and

- (i) f is LS-integrable (ω) on $Q \cap [a_1, b_1]$;
- (ii) if $J_k = [\alpha_k, \beta_k]$ denotes the closed interval contiguous to $Q \cap [a_1, b_1]$ relative to J , then

$$\sum_k |T_0(f, J_k)| < +\infty;$$

- (iii) if x is a limit point of end points of $\{J_k\}$, then there exists a set $A_x \supset Q \cap [a_1, b_1]$ such that A_x has ω^* -density 0 at x and

$$\lim_{k \rightarrow \infty} O(T_0, f, A_x \cap J_k) = 0.$$

Now, $f \in \mathcal{F}(T_0, J_k)$ and $(\alpha_k, \beta_k) \cap Q = \emptyset$ for each k . Hence, by Theorem 4.1, it follows that, for each k ,

$$T_0(f, J_k) = T(f, J_k).$$

Therefore, by $(H\omega)$ property of T and T_0 , we have $f \in \mathcal{F}(T, I)$ and

$$\begin{aligned} T(f, I) &= (\text{LS}) \int_{Q \cap [a_1, b_1]} f d\omega + \sum_k T \int_{J_k \cap I} f d\omega - \\ &\quad - \frac{1}{2} \sum_{\alpha_k \in I} \omega_f(\alpha_k) - \frac{1}{2} \sum_{\beta_k \in I} \omega_f(\beta_k) = T_0(f, I). \end{aligned}$$

The equality $T(f, I) = T_0(f, I)$ is obviously true for any $I' \subset I$ and $I' \in \mathcal{G}$. It follows, therefore, that $T \supset T_0$ on $[a, b]$. Thus we have arrived to a contradiction, which establishes the theorem.

Concluding remarks. Theorem 4.2 shows that the $\overline{\text{ADS}}$ -integral (ω) is the least general among the ω -approximately continuous integrals having properties $(C\omega)$ and $(H\omega)$. If we take $\omega(x) = x$, then the results obtained here coincide with those concerning the AD-integrals of Kubota (see [7] and [9]). Theorem 2.4 is a generalization of Lemma 2.2 in [7]. It appears that condition (H) in Lemma 2 in [9] and the condition “ T includes the L-integral” in Theorem 4 in [9] are superfluous. Indeed, the condition “ T has property $(H\omega)$ ” implies “ T includes the L_ω -integral”. A perusal of the proof of Theorem 3.11 should convince every one that hypothesis (ii) can be replaced by the weaker hypothesis (ii') obtained by replacing condition (*) by the condition

$$\lim_{\alpha_k \rightarrow x} O(\overline{\omega\text{-ADS}}, f, E_x \cap J_k) = 0.$$

Making an analogous alteration in condition (ii) of property $(H\omega)$ in Definition 4.3, we get a stronger property $(H'\omega)$. In the statement of Theorem 4.2, replacing $(H\omega)$ by $(H'\omega)$, we get a true and stronger proposition.

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REFERENCES

- [1] P. C. Bhakta, *On functions of bounded ω -variation*, Rivista di Matematica della Università di Parma (2) 6 (1965), p. 55-64.
- [2] M. C. Chakrabarty, *Some results on AC - ω functions*, Fundamenta Mathematicae 64 (1969), p. 219-230.
- [3] — *Some results on ω -derivatives and BV - ω functions*, Journal of the Australian Mathematical Society 9 (1969), p. 345-360.
- [4] E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, Vol. I, Dover.
- [5] R. L. Jeffery, *Generalized integrals with respect to functions of bounded variation*, Canadian Journal of Mathematics 10 (1958), p. 617-626.
- [6] — *The theory of functions of a real variable*, Toronto 1962.
- [7] Yoto Kubota, *On the approximately continuous Denjoy integrals*, Tohoku Mathematical Journal 15 (1963), p. 251-264.
- [8] — *An integral of the Denjoy type*, Proceedings of the Japan Academy 40 (1964), p. 713-717.
- [9] — *A characterization of the approximately continuous Denjoy integral*, Canadian Journal of Mathematics 22 (1970), p. 219-226.
- [10] M. E. Munroe, *Introduction to measure and integration*, Wesley 1959.
- [11] J. Ridder, *Über Perron-Stieltjessche und Denjoy-Stieltjessche Integrationen*, Mathematische Zeitschrift 40 (1936), p. 127-160.
- [12] S. Saks, *Theory of the integral*, Warszawa 1937.

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